

Appendix B

Vector Analysis Relations

B.1 Vector analysis. Important relations and proofs

B.1.1 Basic relations

B.1.1.1 Divergence of the gradient of a scalar field

$$\begin{aligned}
\operatorname{div} \left[\vec{\operatorname{grad}} a(\vec{r}) \right] &= \sum_{i=1}^3 \vec{e}_i \frac{\partial}{\partial x_i} \cdot \left(\sum_{j=1}^3 \vec{e}_j \frac{\partial}{\partial x_j} a(\vec{r}) \right) \\
&= \sum_{i,j=1}^3 \vec{e}_i \cdot \vec{e}_j \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} a(\vec{r}) \\
&= \sum_{i,j=1}^3 \delta_{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} a(\vec{r}) \\
&= \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2} a(\vec{r}) \\
&= \Delta a(\vec{r})
\end{aligned} \tag{B.1}$$

where $\Delta = \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator.

B.1.1.2 Divergence of the product of a scalar and a vector field

$$\begin{aligned}
\operatorname{div} \left[a(\vec{r}) \vec{b}(\vec{r}) \right] &= \sum_{i=1}^3 \vec{e}_i \cdot \frac{\partial}{\partial x_i} \left[a(\vec{r}) \sum_{j=1}^3 \vec{e}_j b_j(\vec{r}) \right] \\
&= \sum_{i,j=1}^3 \delta_{ij} \frac{\partial}{\partial x_i} [a(\vec{r}) b_j(\vec{r})] \\
&= \sum_{i=1}^3 \left[\frac{\partial a(\vec{r})}{\partial x_i} b_i(\vec{r}) + a(\vec{r}) \frac{\partial b_i(\vec{r})}{\partial x_i} \right] \\
&= a(\vec{r}) \operatorname{div} \vec{b}(\vec{r}) + \vec{b}(\vec{r}) \cdot \vec{\operatorname{grad}} a(\vec{r})
\end{aligned} \tag{B.2}$$

B.1.2 Relations involving vector products

When more complicated mathematical expressions have to be handled, it is a powerful technique to express the vector product by using the Levi-Civita symbol, ε ,

defined as

$$\varepsilon_{ijk} = \left\{ \begin{array}{ll} +1 & \text{if } i, j, k \text{ even permutation} \\ -1 & \text{if } i, j, k \text{ odd permutation} \\ 0 & \text{in all other cases.} \end{array} \right\} \forall i, j, k \in \{1, 2, 3\} \quad (\text{B.3})$$

A basic application of Eq. (B.3) is in the definition of the vector product:

$$\vec{a} \times \vec{b} = \sum_{i,j,k=1}^3 \varepsilon_{ijk} a_i b_j \vec{e}_k \quad (\text{B.4})$$

Similarly, the curl of a vector field can be expressed in a very compact form as

$$\text{rot} \vec{G}(\vec{x}) = \nabla_{\vec{x}} \times \vec{G}(\vec{x}) = \sum_{i,j,k=1}^3 \varepsilon_{ijk} \frac{\partial}{\partial x_i} G_j(\vec{x}) \vec{e}_k \quad (\text{B.5})$$

Suppose we are confronted with an expression, actually occurring frequently in electrodynamics, like $\text{div} [\vec{a} \times \vec{b}]$, where $\vec{a} = \vec{a}(\vec{r})$ and $\vec{b} = \vec{b}(\vec{r})$ are vector fields. With the help of Eq. (B.3) we can write the divergence of the vector product mentioned

above in a very convenient, index-based fashion

$$\begin{aligned}
\operatorname{div} \left[\vec{a}(\vec{r}) \times \vec{b}(\vec{r}) \right] &= \sum_{i=1}^3 \vec{e}_i \cdot \frac{\partial}{\partial x_i} \left[\sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \varepsilon_{jkl} a_k(\vec{r}) b_l(\vec{r}) \vec{e}_j \right] \\
&= \sum_{i,j,k,l=1}^3 \vec{e}_i \cdot \varepsilon_{jkl} \left(\frac{\partial a_k}{\partial x_i} b_l + a_k \frac{\partial b_l}{\partial x_i} \right) \vec{e}_j \\
&= \sum_{i,j,k,l=1}^3 \delta_{ij} \varepsilon_{jkl} \left(\frac{\partial a_k}{\partial x_i} b_l + a_k \frac{\partial b_l}{\partial x_i} \right) \\
&= \sum_{j,k,l=1}^3 \varepsilon_{jkl} \frac{\partial a_k}{\partial x_j} b_l + \sum_{j,k,l=1}^3 \varepsilon_{jkl} a_k \frac{\partial b_l}{\partial x_j} \\
&= \sum_{j,k,l=1}^3 \varepsilon_{klj} \frac{\partial a_l}{\partial x_k} b_j + \sum_{j,k,l=1}^3 \varepsilon_{kjl} a_j \frac{\partial b_l}{\partial x_k} \\
&= \sum_{i,j,k,l=1}^3 \vec{e}_i \cdot \vec{e}_j \varepsilon_{klj} \frac{\partial a_l}{\partial x_k} b_i + \sum_{i,j,k,l=1}^3 \vec{e}_i \cdot \vec{e}_j \varepsilon_{kjl} a_i \frac{\partial b_l}{\partial x_k} \\
&= \sum_i b_i \vec{e}_i \cdot \sum_{j,k,l=1}^3 \varepsilon_{klj} \frac{\partial a_l}{\partial x_k} \vec{e}_j + \sum_i a_i \vec{e}_i \cdot \sum_{j,k,l=1}^3 \varepsilon_{kjl} \frac{\partial b_l}{\partial x_k} \vec{e}_j \\
\operatorname{div} \left[\vec{a} \times \vec{b} \right] &= \vec{b} \cdot \operatorname{rot} \vec{a} - \vec{a} \cdot \operatorname{rot} \vec{b} \tag{B.6}
\end{aligned}$$

The curl of the gradient of an arbitrary scalar field vanishes:

$$\begin{aligned}
\vec{\text{rot}} \left(\vec{\text{grad}} f(\vec{x}) \right) &= \sum_{i,j,k=1}^3 \varepsilon_{ijk} \frac{\partial}{\partial x_i} \left(\sum_{l=1}^3 \vec{e}_l \frac{\partial}{\partial x_l} f(\vec{x}) \right) \cdot \vec{e}_j \vec{e}_k \\
&= \sum_{i,j,k,l=1}^3 \varepsilon_{ijk} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_l} f(\vec{x}) \delta_{jl} \vec{e}_k \\
&= \sum_{i,j,k=1}^3 \varepsilon_{ijk} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} f(\vec{x}) \vec{e}_k \\
&= \left(\frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_1} \right) f(\vec{x}) \vec{e}_3 \\
&\quad \left(\frac{\partial}{\partial x_2} \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_3} \frac{\partial}{\partial x_2} \right) f(\vec{x}) \vec{e}_1 \\
&\quad \left(\frac{\partial}{\partial x_3} \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_3} \right) f(\vec{x}) \vec{e}_2 \\
\vec{\text{rot}} \left(\vec{\text{grad}} f(\vec{x}) \right) &= \vec{0}
\end{aligned} \tag{B.7}$$

Divergence of the curl of an arbitrary vector field:

$$\begin{aligned}
\text{div} \left(\vec{\text{rot}} \vec{G}(\vec{r}) \right) &= \sum_{i=1}^3 \frac{\partial}{\partial x_i} \vec{e}_i \cdot \left(\sum_{j,k,l=1}^3 \varepsilon_{jkl} \frac{\partial}{\partial x_j} G_k(\vec{r}) \vec{e}_l \right) \\
&= \sum_{i,j,k,l=1}^3 \delta_{il} \varepsilon_{jkl} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} G_k(\vec{r}) \\
&= \sum_{i,j,k=1}^3 \varepsilon_{jki} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} G_k(\vec{r}) \\
&= \sum_{i,j,k=1}^3 \varepsilon_{ijk} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} G_k(\vec{r}) \\
&= 0
\end{aligned} \tag{B.8}$$

The last step is evident as for every term k in the sum there is a pair of terms with

$i \neq j$ with opposite signs.

Double curl of an arbitrary vector field:

$$\begin{aligned}
\vec{\text{rot}} \left(\vec{\text{rot}} \vec{G}(\vec{x}) \right) &= \sum_{i,j,k=1}^3 \varepsilon_{ijk} \frac{\partial}{\partial x_i} \left(\sum_{l,m,n=1}^3 \varepsilon_{lmn} \frac{\partial}{\partial x_l} G_m(\vec{x}) \vec{e}_n \right) \cdot \vec{e}_j \vec{e}_k \\
&= \sum_{i,j,k,l,m,n=1}^3 \varepsilon_{ijk} \varepsilon_{lmn} \delta_{jn} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_l} G_m(\vec{x}) \vec{e}_k \\
&= \sum_{i,j,k,l,m=1}^3 \varepsilon_{jki} \varepsilon_{jlm} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_l} G_m(\vec{x}) \vec{e}_k \\
&= \sum_{i,k,l,m=1}^3 (\delta_{kl} \delta_{im} - \delta_{km} \delta_{il}) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_l} G_m(\vec{x}) \vec{e}_k \\
&= \sum_{i,k=1}^3 \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} G_i(\vec{x}) \vec{e}_k - \sum_{i,k=1}^3 \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i} G_k(\vec{x}) \vec{e}_k \\
&= \sum_{k=1}^3 \frac{\partial}{\partial x_k} \vec{e}_k \left(\sum_{i=1}^3 \frac{\partial}{\partial x_i} G_i(\vec{x}) \right) - \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2} \left(\sum_{k=1}^3 G_k(\vec{x}) \vec{e}_k \right) \\
&= \vec{\text{grad}} \left(\text{div} \vec{G}(\vec{x}) \right) - \Delta \vec{G}(\vec{x}) \tag{B.9}
\end{aligned}$$

Here the following theorem has been used:

$$\sum_{j=1}^3 \varepsilon_{jki} \varepsilon_{jlm} = \delta_{kl} \delta_{im} - \delta_{km} \delta_{il} \tag{B.10}$$

which can be verified in an elementary fashion by running over all possible values of

$$\{i, k, l, m\} \in \{1, 2, 3\}.$$

Curl of the product of a scalar and a vector field:

$$\begin{aligned}
\vec{\text{rot}} \left(u(\vec{x}) \vec{G}(\vec{x}) \right) &= \sum_{i,j,k=1}^3 \varepsilon_{ijk} \frac{\partial}{\partial x_i} \left(\sum_{l=1}^3 u(\vec{x}) G_l(\vec{x}) \vec{e}_l \right) \cdot \vec{e}_j \vec{e}_k \\
&= \sum_{i,j,k,l=1}^3 \varepsilon_{ijk} \delta_{jl} \left(u(\vec{x}) \frac{\partial G_l(\vec{x})}{\partial x_i} + \frac{\partial u(\vec{x})}{\partial x_i} G_l(\vec{x}) \right) \vec{e}_k \\
&= \sum_{i,j,k=1}^3 \varepsilon_{ijk} \left(u(\vec{x}) \frac{\partial G_j(\vec{x})}{\partial x_i} + \frac{\partial u(\vec{x})}{\partial x_i} G_j(\vec{x}) \right) \vec{e}_k \\
&= u(\vec{x}) \sum_{i,j,k=1}^3 \varepsilon_{ijk} \frac{\partial}{\partial x_i} G_j(\vec{x}) \vec{e}_k \\
&\quad + \sum_{i,j,k=1}^3 \varepsilon_{ijk} \left(\sum_{l=1}^3 \vec{e}_l \frac{\partial u(\vec{x})}{\partial x_l} \right) \cdot \vec{e}_i G_j(\vec{x}) \vec{e}_k \\
&= u(\vec{x}) \vec{\text{rot}} \vec{G}(\vec{x}) + \left(\vec{\text{grad}} u(\vec{x}) \right) \times \vec{G}(\vec{x}) \tag{B.11}
\end{aligned}$$