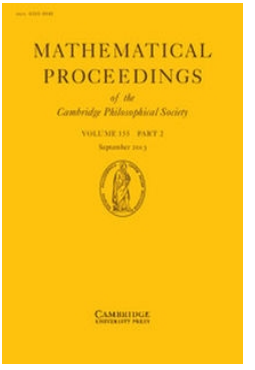

Discussion of the infinite distribution of electrons in the theory of the positron

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Discussion of the infinite distribution of electrons in the theory of the positron. By Professor P. A. M. DIRAC, St John's College.

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1. *Use of the density matrix.*

The quantum theory of the electron allows states of negative kinetic energy as well as the usual states of positive kinetic energy and also allows transitions from one kind of state to the other. Now particles in states of negative kinetic energy are never observed in practice. We can get over this discrepancy between theory and observation by assuming that, in the world as we know it, nearly all the states of negative kinetic energy are occupied, with one electron in each state in accordance with Pauli's exclusion principle, and that the distribution of negative-energy electrons is unobservable to us on account of its uniformity. Any unoccupied negative-energy states would be observable to us, as holes in the distribution of negative-energy electrons, but these holes would appear as particles with positive kinetic energy and thus not as things foreign to all our experience. It seems reasonable and in agreement with all the facts known at present to identify these holes with the recently discovered positrons and thus to obtain a theory of the positron*.

We now have a picture of the world in which there are an infinite number of negative-energy electrons (in fact an infinite number per unit volume) having energies extending continuously from $-mc^2$ to $-\infty$. The problem we have to consider is the way this infinity can be handled mathematically and the physical effects it produces. In particular, we must set up some assumptions for the production of electromagnetic field by the electron distribution, which assumptions must be such that any finite change in the distribution produces a change in the field in agreement with Maxwell's equations, but such that the infinite field which would be required by Maxwell's equations from an infinite density of electrons is in some way cut out.

These problems are quite simple when we suppose each electron to be moving in a space free of electromagnetic field. They are not so simple when there is a field present, since the positive- and negative-energy states then get mixed together so intimately that one cannot in general distinguish accurately between them in a

* As this theory was first put forward, *Proc. Roy. Soc.* A, 126, p. 360 (1930) and *Proc. Camb. Phil. Soc.* 26, p. 361 (1930), the holes were assumed to be protons, but this assumption was afterwards seen to be untenable, since it was found that the holes must correspond to particles with the same rest-mass as electrons. See *Proc. Roy. Soc.* A, 133, p. 61 (1931).

relativistically invariant way. A careful investigation is then necessary, even for such an elementary problem as seeing that a precise meaning can be given to a distribution such as occurs in practice, in which nearly all the negative-energy states are occupied and nearly all the positive-energy ones unoccupied.

To make an exact treatment of the matter would be very complicated and in the present paper only an approximate treatment will be given, on the lines of Hartree's method of the self-consistent field. We shall suppose that each electron has its own individual wave function in space-time (instead of there being one wave function in an enormous number of variables to describe the whole distribution), and also we shall suppose that each electron moves in a definite electromagnetic field, which is the same for all the electrons. This field will consist of a part coming from external causes and a part coming from the electron distribution itself, the precise way in which the latter part depends on the electron distribution being one of the problems we have to consider.

Let the normalized functions for the electrons at any time be $\psi_a(x)$, where x stands for three positional coordinates x_1, x_2, x_3 of an electron and the suffix a takes on different values for the different electrons. With electron spin taken into account, each $\psi_a(x)$ must have four components, which may be specified by $\psi_{ak}(x)$ with $k = 1, 2, 3, 4$. The whole distribution of electrons may now be described by the density matrix ρ defined by

$$(x' \,|\, \rho \,|\, x'')_{k'k''} = \Sigma_a \psi_{ak'}(x') \overline{\psi}_{ak''}(x''), \tag{1}$$

in which the sum is taken over all the electrons. This is a matrix in the spin variables k as well as in the positional variables x. It is, of course, a Hermitian matrix. Its properties have been studied previously*, the chief ones being the equation

$$\rho^2 = \rho, \tag{2}$$

which expresses that the electron distribution satisfies the exclusion principle, and the equation of motion

$$i\hbar \frac{d\rho}{dt} = H\rho - \rho H. \tag{3}$$

Here H is the Hamiltonian for the motion of a single electron in the field, thus

$$H = \alpha_s(p_s + eA_s) - eA_0 + \alpha_4 m, \tag{4}$$

the velocity of light being made equal to unity and a summation being implied over the values $s = 1, 2, 3$.

* Dirac, *Proc. Camb. Phil. Soc.* 25, p. 62 (1929); 26, p. 376 (1930) and 27, p. 240 (1931).

An alternative way of regarding the sum on the right-hand side of (1) is as a sum over all the occupied states*. It may then conveniently be written $\Sigma_{oc}\psi_{k'}(x')\overline{\psi}_{k''}(x'')$. There will be a corresponding sum over the unoccupied states, which may be written $\Sigma_{un}\psi_{k'}(x')\overline{\psi}_{k''}(x'')$. If we add these two sums, we get the sum over all states and this must give us the unit matrix, from the transformation theory of quantum mechanics. Thus

$$\Sigma_{oc}\psi_{k'}(x')\overline{\psi}_{k''}(x'')+\Sigma_{un}\psi_{k'}(x')\overline{\psi}_{k''}(x'')=\delta(x'-x'')\,\delta_{k'k''}.$$

Put
$$\rho=\tfrac{1}{2}(1+\rho_1), \tag{5}$$

so that

$$(x'\,|\,\rho_1\,|\,x'')_{k'k''}=\Sigma_{oc}\psi_{k'}(x')\overline{\psi}_{k''}(x'')-\Sigma_{un}\psi_{k'}(x')\overline{\psi}_{k''}(x'').$$

We may now consider the electron distribution as specified by the matrix ρ_1 instead of the matrix ρ. This has the advantage that it makes a closer symmetry between the electrons and the positrons and leads to neater mathematical expressions. The equation of motion (3) holds unchanged with ρ_1 instead of ρ and equation (2) gets modified to

$$\rho_1^2=1. \tag{6}$$

The density matrices that we have been discussing up to the present are non-relativistic things, since their elements each refer to two points in space x' and x'' but to only one time. To get a relativistic theory, we must introduce two times, t' and t'', and use instead of ρ the relativistic density matrix R defined by

$$\begin{aligned}(x'\,t'\,|\,R\,|\,x''\,t'')_{k'k''}&=\Sigma_a\psi_{ak'}(x'\,t')\overline{\psi}_{ak''}(x''\,t'')\\&=\Sigma_{oc}\psi_{k'}(x'\,t')\overline{\psi}_{k''}(x''\,t'').\end{aligned} \tag{7}$$

Instead of ρ_1 we shall now have R_1, defined by

$$\begin{aligned}&(x'\,t'\,|\,R_1\,|\,x''\,t'')_{k'k''}\\&\quad=\Sigma_{oc}\psi_{k'}(x'\,t')\overline{\psi}_{k''}(x''\,t'')-\Sigma_{un}\psi_{k'}(x'\,t')\overline{\psi}_{k''}(x''\,t''),\end{aligned}$$

and instead of equation (5) we shall have

$$R=\tfrac{1}{2}(R_F+R_1),$$

where $(x'\,t'\,|\,R_F\,|\,x''\,t'')_{k'k''}$

$$=\Sigma_{oc}\psi_{k'}(x'\,t')\overline{\psi}_{k''}(x''\,t'')+\Sigma_{un}\psi_{k'}(x'\,t')\overline{\psi}_{k''}(x''\,t'').$$

R_F, representing the full distribution with all possible states occupied, is no longer simply the unit matrix, but all the same we should expect it to play some fundamental part in the theory.

* The word 'all' used in this connection means each of a set of orthogonal states which is made as large as possible, and does not include states formed by superposition of these orthogonal states.

The new matrices R, R_1, R_F are also Hermitian and their equation of motion is

$$\mathscr{H}R=0,\ \mathscr{H}R_1=0,\ \mathscr{H}R_F=0, \tag{8}$$

where $\mathscr{H}$ is the total operator that operates on the wave function in the wave equation for one electron, i.e.

$$\mathscr{H}=W-H,$$

W being the operator $i\hbar$ times time-differentiation. Equations (2) and (6) cannot be concisely expressed in terms of the R's.

To obtain the field produced by the distribution of electrons, we must first get the electric density and current density. For this purpose we must, according to the usual theory for finite distributions, take a diagonal element of ρ in the positional variables, or a diagonal element of R in the positional and time variables, and form its diagonal sum over the spin variables. The resulting expression, namely $\Sigma_k(x|\rho|x)_{kk}$ or $\Sigma_k(xt|R|xt)_{kk}$ would then be the electric density (apart from the factor $-e$). The corresponding current density would have for its sth component $\Sigma_k(x|\alpha_s\rho|x)_{kk}$ or $\Sigma_k(xt|\alpha_s R|xt)_{kk}$. We can easily verify that this electric density and current density satisfy the conservation law of electricity. In equation (3) let us take diagonal elements in the positional variables, but keep to the symbolic matrix notation for the spin variables, so that a symbol like $(x|\rho|x)$ denotes a matrix with four rows and columns, of the same nature as an α. This gives

$$\begin{aligned} i\hbar\frac{d}{dt}(x|\rho|x) &= (x|H\rho-\rho H|x) \\ &= \alpha_s(x|p_s\rho-\rho p_s|x) \\ &\quad +(x|\{\alpha_s\rho-\rho\alpha_s\}\{p_s+eA_s\}|x)+(x|\alpha_4\rho-\rho\alpha_4|x)\,m. \end{aligned}$$

If we now take the diagonal sum with respect to the spin variables, the last two terms will contribute nothing, from the rule that the diagonal sum of the product of two matrices is independent of their order, and we shall be left with

$$\begin{aligned} i\hbar\frac{d}{dt}\Sigma_k(x|\rho|x)_{kk} &= \Sigma_{kk'}\alpha_{skk'}(x|p_s\rho-\rho p_s|x)_{k'k} \\ &= -i\hbar\,\Sigma_{kk'}\alpha_{skk'}\frac{\partial}{\partial x_s}(x|\rho|x)_{k'k}, \end{aligned}$$

i.e.

$$\frac{d}{dt}\Sigma_k(x|\rho|x)_{kk}=-\frac{\partial}{\partial x_s}\Sigma_k(x|\alpha_s\rho|x)_{kk}, \tag{9}$$

which is the required conservation law.

In our present theory the electric density and current density given by these formulae would be infinite and some alteration of the assumptions is therefore necessary. The problem now presents itself of finding some natural way of removing the infinities from

$\Sigma_k (xt \mid R \mid xt)_{kk}$ and $\Sigma_k (xt \mid \alpha_s R \mid xt)_{kk}$ so as to leave finite remainders, which we could then assume to be the electric and current densities. This problem requires us to make a detailed investigation of the singularities in the matrix elements $(x'\,t' \mid R \mid x''\,t'')_{k'k''}$ near the diagonal $x_s' = x_s''$, $t' = t''$.

2. *Case of no field.*

We shall begin our investigation by taking the case of no electromagnetic field, when the Hamiltonian (4) reduces to

$$H = \alpha_s p_s + \alpha_4 m = (\boldsymbol{\alpha}, \mathbf{p}) + \alpha_4 m. \tag{10}$$

In this case we can calculate accurately the matrix elements $(x'\,t' \mid R \mid x''\,t'')_{k'k''}$ for the distribution of electrons in which all the negative-energy states are occupied and all the positive-energy ones unoccupied, and see exactly how these matrix elements behave near the diagonal.

If we try to work directly from the definition (7) we meet with some awkward calculations in taking the spin variables into account and summing over the two possible spin orientations. We can avoid these calculations by using symbolic methods and first obtaining ρ. The condition that a wave function ψ contains only Fourier components belonging to negative-energy states may be expressed symbolically by

$$\{H + \sqrt{(P^2 + m^2)}\}\,\psi = 0,$$

where P denotes the length of the vector $\mathbf{p}$ and the positive square root is understood. Similarly, the condition that the distribution ρ contains electrons only in negative-energy states may be expressed by

$$\{H + \sqrt{(P^2 + m^2)}\}\,\rho = 0. \tag{11}$$

The condition that in the distribution ρ every negative-energy state is occupied, is just the condition that the distribution $1 - \rho$ contains electrons only in positive-energy states and may thus be expressed by

$$\{H - \sqrt{(P^2 + m^2)}\}\,(1 - \rho) = 0.$$

Adding this equation to (11) we get

$$\{H + \sqrt{(P^2 + m^2)}\}\,(2\rho - 1) = 0$$

or
$$\rho = \frac{1}{2}\left[1 - \frac{H}{\sqrt{(P^2 + m^2)}}\right] = \frac{1}{2}\left[1 - \frac{(\boldsymbol{\alpha}, \mathbf{p}) + \alpha_4 m}{\sqrt{(P^2 + m^2)}}\right]. \tag{12}$$

Hence, from the transformation theory,

$$(x' \mid \rho \mid x'') = \frac{1}{2h^3}\iint e^{i(\mathbf{x}', \mathbf{p}')/\hbar}\,dp'\left[1 - \frac{(\boldsymbol{\alpha}, \mathbf{p}') + \alpha_4 m}{\sqrt{(P'^2 + m^2)}}\right] \times \delta(p' - p'')\,dp''\,e^{-i(\mathbf{x}'', \mathbf{p}')/\hbar}, \tag{13}$$

where dp denotes the product $dp_1 dp_2 dp_3$.

We can now easily see that

$$(x't' \mid R \mid x''t'')$$
$$= \frac{1}{2h^3}\iint e^{i(\mathbf{x}',\mathbf{p}')/\hbar}\, e^{it'\sqrt{(P'^2+m^2)}/\hbar}\, dp' \left[1 - \frac{(\boldsymbol{\alpha}, \mathbf{p}')+\alpha_4 m}{\sqrt{(P'^2+m^2)}}\right]$$
$$\times \delta(p'-p'')\, dp''\, e^{-i(\mathbf{x}'',\mathbf{p}'')/\hbar}\, e^{-it''\sqrt{(P''^2+m^2)}/\hbar}. \quad (14)$$

This is because the right-hand side of (14) is Hermitian and goes over into the right-hand side of (13) when $t' = t''$, and also it satisfies the equation of motion (8), since the operator $\mathscr{H}$ operating on the integrand in (14) is equivalent to the factor

$$-\{(\boldsymbol{\alpha}, \mathbf{p}') + \alpha_4 m + \sqrt{(P'^2+m^2)}\}$$

multiplied into this integrand on the left and therefore gives the result zero, from equation (11). Introducing the notation

$$x_s' - x_s'' = x_s, \quad t' - t'' = t,$$

which we shall keep through the rest of the paper, we get from (14)

$$(x't' \mid R \mid x''t'') = \frac{1}{2h^3}\int e^{i(\mathbf{x},\mathbf{p})/\hbar}\, e^{it\sqrt{(P^2+m^2)}/\hbar}$$
$$\times \left[1 - \frac{(\boldsymbol{\alpha}, \mathbf{p}) + \alpha_4 m}{\sqrt{(P^2+m^2)}}\right] dp$$
$$= -\left[i\hbar\frac{\partial}{\partial t} - i\hbar\alpha_s\frac{\partial}{\partial x_s} + \alpha_4 m\right] S(x,t), \quad (15)$$

where $$S(x,t) = \frac{1}{2h^3}\int e^{i(\mathbf{x},\mathbf{p})/\hbar}\, e^{it\sqrt{(P^2+m^2)}/\hbar}\, \frac{dp}{\sqrt{(P^2+m^2)}}.$$

To integrate this expression for S, let r denote the length of the vector $\mathbf{x}$ and θ the angle between it and the vector $\mathbf{p}$. Then

$$S(x,t) = \frac{\pi}{h^3}\int_0^\infty P^2 dP \int_{-\pi}^{\pi} \sin\theta\, d\theta\, e^{irP\cos\theta/\hbar}\, e^{it\sqrt{(P^2+m^2)}/\hbar}$$
$$\div \sqrt{(P^2+m^2)}$$
$$= \frac{-i}{2h^2 r}\int_0^\infty \{e^{i[rP + t\sqrt{(P^2+m^2)}]/\hbar} - e^{i[-rP + t\sqrt{(P^2+m^2)}]/\hbar}\}\, P\, dP$$
$$\div \sqrt{(P^2+m^2)}$$
$$= \frac{-i}{2h^2 r}\int_{-\infty}^{\infty} e^{i[rP + t\sqrt{(P^2+m^2)}]/\hbar}\, P\, dP/\sqrt{(P^2+m^2)}$$
$$= \frac{-i}{4h}\frac{1}{r}\frac{\partial}{\partial r} U(r,t), \quad (16)$$

where

$$U(r,t) = -\frac{i}{\pi}\int_{-\infty}^{\infty} e^{i[rP + t\sqrt{(P^2+m^2)}]/\hbar}\, dP/\sqrt{(P^2+m^2)}$$

$$= -\frac{i}{\pi}\int_{-\infty}^{\infty} \exp\{i[r\sinh\chi + t\cosh\chi]\, m/\hbar\}\, d\chi,$$

with $\quad P = m\sinh\chi, \qquad \sqrt{(P^2+m^2)} = m\cosh\chi.$

This integral can be evaluated in terms of Bessel functions and the result is

$$\begin{aligned} U(r,t) &= H_0^{(1)}\{m(t^2-r^2)^{\frac{1}{2}}/\hbar\} && t > r \\ &= H_0^{(1)}\{im(r^2-t^2)^{\frac{1}{2}}/\hbar\} && r > t > -r \\ &= -H_0^{(2)}\{m(t^2-r^2)^{\frac{1}{2}}/\hbar\} && -r > t. \end{aligned}$$

If we went through the corresponding calculation for the distribution of electrons in which all the negative-energy states are unoccupied and all the positive-energy ones occupied, we should get a result of the form (15), where S would be given by equation (16) with $-U(r,-t)$ substituted for $U(r,t)$. Hence the full distribution R_F, with all the positive and negative-energy states occupied, will be given by

$$(x't' \,|\, R_F \,|\, x''t'') = -\left[i\hbar\frac{\partial}{\partial t} - i\hbar\alpha_s\frac{\partial}{\partial x_s} + \alpha_4 m\right] S_F(x,t), \quad (17)$$

$$S_F(x,t) = \frac{-i}{4h}\frac{1}{r}\frac{\partial}{\partial r}U_F(r,t), \quad (18)$$

with $\quad U_F(r,t) = U(r,t) - U(r,-t)$

$$\left.\begin{aligned} &= 2J_0\{m(t^2-r^2)^{\frac{1}{2}}/\hbar\} && t > r \\ &= 0 && r > t > -r \\ &= -2J_0\{m(t^2-r^2)^{\frac{1}{2}}/\hbar\} && -r > t \end{aligned}\right\}. \quad (19)$$

Similarly the distribution R_1 will be given by

$$(x't' \,|\, R_1 \,|\, x''t'') = -\left[i\hbar\frac{\partial}{\partial t} - i\hbar\alpha_s\frac{\partial}{\partial x_s} + \alpha_4 m\right] S_1(x,t), \quad (20)$$

$$S_1(x,t) = \frac{-i}{4h}\frac{1}{r}\frac{\partial}{\partial r}U_1(r,t), \quad (21)$$

with $\quad U_1(r,t) = U(r,t) + U(r,-t)$

$$\left.\begin{aligned} &= 2iY_0\{m(t^2-r^2)^{\frac{1}{2}}/\hbar\} && t > r \\ &= 2H_0^{(1)}\{im(r^2-t^2)^{\frac{1}{2}}/\hbar\} && r > t > -r \\ &= 2iY_0\{m(t^2-r^2)^{\frac{1}{2}}/\hbar\} && -r > t \end{aligned}\right\}. \quad (22)$$

It is clear from these equations that there will be singularities, not only at the point $x_s = 0, t = 0$, but also everywhere on the light-cone $t^2 - r^2 = 0$. In order to determine these singularities, we may expand the Bessel functions in power series of $\sqrt{(t^2 - r^2)}$ and retain only the first few terms. If we retain only the first term in (19), we get for U_F the constant value zero outside the light-cone and the constant values 2, -2 in the two regions inside the light-cone for which $t > r$ and $t < -r$ respectively. This U_F substituted in (18) gives

$$S_F(x,t) = (i/2hr)\{\delta(r-t) - \delta(r+t)\}.$$

For $t > 0$, this may be written

$$S_F(x,t) = (i/2hr)\,\delta(r-t) = (i/h)\,\delta(t^2 - r^2),$$

and when substituted in (17) gives

$$(x't' \mid R_F \mid x''t'') = (1/\pi)(t + a_s x_s)\,\delta'(t^2 - r^2), \qquad (23)$$

with neglect of a term involving $\delta(t^2 - r^2)$. This is the worst singularity of R_F. For $t < 0$ the result is the same, except for a change of sign. If we retained the second term in the expansion of J_0 in (19), we should get the next worst singularity of R_F, containing $\delta(t^2 - r^2)$. If we retained also the third term in J_0 in (19), we should get also the third worst kind of singularity in R_F, involving a plain discontinuity on the light-cone. The fourth and higher terms in J_0 would not give rise to any singularity in R_F. In this way we can determine completely all the singularities in R_F.

Let us now examine the singularities in R_1. The important terms in U_1, given by (22), are

$$(4i/\pi)\log\{m(t^2 - r^2)^{\frac{1}{2}}/\hbar\}\,J_0\{m(t^2 - r^2)^{\frac{1}{2}}/\hbar\} \qquad |t| > r, \quad (24)$$

$$[2 + (4i/\pi)\log\{im(r^2 - t^2)^{\frac{1}{2}}/\hbar\}]\,J_0\{im(r^2 - t^2)^{\frac{1}{2}}/\hbar\} \qquad |t| < r. \quad (25)$$

The remaining terms are a power series in $t^2 - r^2$, of the same form inside and outside the light-cone, and therefore they do not give rise to any singularity. Expression (25) may be simplified to

$$(4i/\pi)\log\{m(r^2 - t^2)^{\frac{1}{2}}/\hbar\}\,J_0\{im(r^2 - t^2)^{\frac{1}{2}}/\hbar\}. \qquad |t| < r.$$

Substituting this and (24) into (21) we get, if we take only the first term of J_0 into account,

$$S_1(x,t) = \frac{1}{\pi h}\frac{1}{r^2 - t^2}. \qquad (26)$$

It is important to see that no δ function occurs in $S_1(x,t)$. The reason at the bottom of this is that, in differentiating the logarithm that occurs in the Bessel functions Y_0 and $H_0^{(1)}$, we must use the formula

$$\frac{d}{dz}\log z = \frac{1}{z} - i\pi\,\delta(z) \qquad (27)$$

in which the term $-i\pi\delta(z)$ is required in order to make the integral of the right-hand side of (27) between the limits a and $-a$ equal $\log -1$, the integral of $1/z$ between these limits being assumed to be zero. The δ function which arises in this way just cancels with that arising from the fact that we have $H_0^{(1)}$ instead of Y_0 in (22) when $r > t > -r$. This cancellation is exact and still holds when (22) is differentiated more than once with respect to any of the variables t, r, x_s.

On substituting (26) into (20), we find

$$(x't' \mid R_1 \mid x''t'') = \frac{-i}{\pi^2} \frac{t + \alpha_s x_s}{(t^2 - r^2)^2}, \tag{28}$$

neglecting a term involving $1/(t^2 - r^2)$. This gives the worst singularity of R_1. By taking into account the second and third terms in the expansion of the J_0 in (24) and (25), we can calculate the other singularities in R_1, involving $1/(t^2 - r^2)$ and $\log|t^2 - r^2|$.

The main result of this investigation for the case of no field is that there are two quite distinct kinds of singularity occurring in the matrices R_F and R_1 respectively. The singularities occurring in R_F are all associated with the δ function and those in R_1 with the reciprocal function and logarithm. From the generality of this result we may expect it to hold also when there is a field present.

3. *Case of an arbitrary field.*

Let us now examine the singularities in $(x't' \mid R_F \mid x''t'')$ and $(x't' \mid R_1 \mid x''t'')$ when there is a general field present. Our method will be to suppose that the singularities are of the same form as in the case of no field, but have unknown coefficients. These coefficients must be functions of x_s', t', x_s'', t'' which are free from singularities and can be expanded as Taylor series for small values of x_s and t. We must try to choose them so that the equations of motion (8) are satisfied.

The application of the method follows a parallel course for R_F and R_1, and we need therefore treat in detail only R_1, which is the density matrix we are mainly interested in. We put

$$(x't' \mid R_1 \mid x''t'') = u \frac{t + \alpha_s x_s}{(t^2 - r^2)^2} + \frac{v}{t^2 - r^2} + w \log|t^2 - r^2|, \tag{29}$$

where u, v and w are functions of x_s', t', x_s'', t'' or of x_s, t, x_s'', t'', which are free from singularities for small values of x_s and t. To get sufficient generality we must allow u, v and w to be matrices in the spin variables and thus not necessarily to commute with the α's. We shall find, however, that we can satisfy all the conditions with u diagonal in the spin variables and thus commuting with all the α's. For the sake of brevity in the algebraic work, we shall assume already now that u is diagonal in the spin variables.

We must now try to choose u, v and w so that the second of equations (8) is satisfied. It is convenient to use the symbol $\mathscr{H}$ to denote the differential operator

$$\mathscr{H} = i\hbar\left(\frac{\partial}{\partial t} + \alpha_s \frac{\partial}{\partial x_s}\right) + e(A_0 - \alpha_s A_s) - \alpha_4 m, \tag{30}$$

the function that it operates on being assumed to be expressed in terms of the variables x_s, t, x_s'', t'' and not to involve x_s', t' explicitly. With this notation, we get from (8)

$$\mathscr{H}\left\{u\frac{t+\alpha_s x_s}{(t^2-r^2)^2} + \frac{v}{t^2-r^2} + w\log|t^2-r^2|\right\} = 0,$$

which reduces to

$$(\mathscr{H}u)\frac{t+\alpha_s x_s}{(t^2-r^2)^2} + (\mathscr{H}v)\frac{1}{t^2-r^2} - 2i\hbar\frac{t-\alpha_s x_s}{(t^2-r^2)^2}v + (\mathscr{H}w)\log|t^2-r^2| + 2i\hbar\frac{t-\alpha_s x_s}{t^2-r^2}w = 0. \tag{31}$$

In order that this equation may hold when u, v and w are free from singularity, the log term must vanish by itself and hence

$$\mathscr{H}w = 0 \tag{32}$$

and

$$(\mathscr{H}u)(t+\alpha_s x_s) + (\mathscr{H}v)(t^2-r^2) - 2i\hbar(t-\alpha_s x_s)v + 2i\hbar(t-\alpha_s x_s)w(t^2-r^2) = 0. \tag{33}$$

Now from the commutability relations for the α's we find, after a simple calculation,

$$(\mathscr{H}u)(t+\alpha_s x_s) = -(t-\alpha_s x_s)\mathscr{G}u + 2\left\{t\left(i\hbar\frac{\partial}{\partial t} + eA_0\right) + x_s\left(i\hbar\frac{\partial}{\partial x_s} - eA_s\right)\right\}u, \tag{34}$$

where $\mathscr{G}$ denotes the differential operator

$$\mathscr{G} = i\hbar\left(\frac{\partial}{\partial t} - \alpha_s\frac{\partial}{\partial x_s}\right) + e(A_0 + \alpha_s A_s) + \alpha_4 m. \tag{35}$$

If we substitute the right-hand side of (34) for the first term in (33), we get an equation which may be written

$$2\left\{t\left(i\hbar\frac{\partial}{\partial t} + eA_0\right) + x_s\left(i\hbar\frac{\partial}{\partial x_s} - eA_s\right)\right\}u + (t-\alpha_s x_s)B = 0, \tag{36}$$

where

$$B = (t+\alpha_s x_s)\mathscr{H}v - 2i\hbar v + 2i\hbar w(t^2-r^2) - \mathscr{G}u. \tag{37}$$

When a given $(x't'|R_1|x''t'')$ is expressed in the form (29), u and v are not completely determined, since we can always add a term of the form $b(t-\alpha_s x_s)$ to u and subtract b from v without changing the right-hand side of (29), b being anything free from

singularities. It can easily be shown that we may choose b so as to make $B = 0$. With this extra condition, equation (36) can be solved for u, the result being

$$u = k \exp \{ie \int (A_0 dt - A_s dx_s)/\hbar\}, \tag{38}$$

where k is an arbitrary coefficient and the integral is taken along the straight line in space-time joining the point x_s'', t'' to the point x_s', t'. We must take k equal to $-i/\pi^2$ so as to make the worst singularity of the right-hand side of (29) equal to the right-hand side of (28) for small values of x_s, t. This determines u completely.

We must now deal with equation (37) with $B = 0$. It may be written

$$(t + \alpha_s x_s) f - 2i\hbar v - \mathscr{G} u = 0, \tag{39}$$

where

$$f = \mathscr{H} v + 2i\hbar (t - \alpha_s x_s) w. \tag{40}$$

Equation (39) gives v in terms of f and allows us to eliminate v and work with the unknown f instead. Eliminating v from (40), we get

$$2i\hbar f = \mathscr{H}(t + \alpha_s x_s) f - \mathscr{H}\mathscr{G} u - 4\hbar^2 (t - \alpha_s x_s) w. \tag{41}$$

Now by the same kind of calculation as led to (34) we find

$$\mathscr{H}(t + \alpha_s x_s) f = -(t - \alpha_s x_s)\, \mathscr{G} f + 4i\hbar f + 2 \left\{ t \left(i\hbar \frac{\partial}{\partial t} + eA_0 \right) + x_s \left(i\hbar \frac{\partial}{\partial x_s} - eA_s \right) \right\} f. \tag{42}$$

If this expression is substituted for the first term in the right-hand side of (41), we get the result

$$2 \left\{ t \left(i\hbar \frac{\partial}{\partial t} + eA_0 \right) + x_s \left(i\hbar \frac{\partial}{\partial x_s} - eA_s \right) \right\} f + 2i\hbar f = \mathscr{H}\mathscr{G} u + (t - \alpha_s x_s)(4\hbar^2 w + \mathscr{G} f). \tag{43}$$

A way of solving this equation for f is first to solve the corresponding equation with the term containing the factor $t - \alpha_s x_s$ omitted, i.e. the equation

$$2 \left\{ t \left(i\hbar \frac{\partial}{\partial t} + eA_0 \right) + x_s \left(i\hbar \frac{\partial}{\partial x_s} - eA_s \right) \right\} f_1 + 2i\hbar f_1 = \mathscr{H}\mathscr{G} u. \tag{44}$$

Then f will be of the form

$$f = f_1 + (t - \alpha_s x_s) g, \tag{45}$$

where g is free from singularities. Substituting (45) in (43) and using (44), we get, after cancelling the factor $t - \alpha_s x_s$,

$$2 \left\{ t \left(i\hbar \frac{\partial}{\partial t} + eA_0 \right) + x_s \left(i\hbar \frac{\partial}{\partial x_s} - eA_s \right) \right\} g + 4i\hbar g = 4\hbar^2 w + \mathscr{G} f_1 + \mathscr{G} (t - \alpha_s x_s) g.$$

If we now use an equation like (42), with g instead of f and the signs of all the α's changed, this reduces to

$$(t+\alpha_s x_s)\,\mathscr{H}g = 4\hbar^2 w + \mathscr{S}f_1. \tag{46}$$

Equation (44) fixes f_1 completely, without any arbitrary constant. One can see this by substituting for the various functions in this equation their Taylor expansions in powers of x_s and t, when it will be found that the coefficients of f_1 of the nth degree are completely determined in terms of those of lower degree and of the coefficients of the A's.

We are thus left with the problem of solving equations (32) and (46) for w and g. If we apply the operator $\mathscr{H}$ to equation (46), the term involving w drops out by equation (32), leaving

$$\mathscr{H}(t+\alpha_s x_s)\,\mathscr{H}g = \mathscr{H}\mathscr{S}f_1. \tag{47}$$

This, considered as an equation in the unknown $\mathscr{H}g$, determines $\mathscr{H}g$ completely, as may also be verified by Taylor expansions. From (46), w is now determined. In this way all our equations are satisfied and all our unknowns are determined, with the exception of g, which is not itself determined although $\mathscr{H}g$ is. The final result is

$$(x't'\,|\,R_1\,|\,x''t'') = u\,\frac{t+\alpha_s x_s}{(t^2-r^2)^2} + \frac{(t+\alpha_s x_s)\,f_1 - \mathscr{S}u}{2i\hbar\,(t^2-r^2)} + \frac{g}{2i\hbar} + w\log|t^2-r^2|, \tag{48}$$

where u is given by (38), and f_1, $\mathscr{H}g$ and w are determined by (44), (47) and (46) respectively.

To do the corresponding work for R_F we put, analogously to (29),

$$(x't'\,|\,R_F\,|\,x''t'') = u\,(t+\alpha_s x_s)\,\delta'(t^2-r^2) + v\delta(t^2-r^2) + w\gamma(t^2-r^2), \tag{49}$$

where γ is the function

$$\gamma(z)=0 \qquad z<0$$
$$\gamma(z)=1 \qquad z>0,$$

and we again try to choose u, v and w to satisfy the equation of motion (8). The equations that we now get for u, v and w are exactly the same as before and thus their solution will be the same, or else will differ from the previous solution by a numerical factor. In order that the worst singularity of the right-hand side of (49), i.e. the first term in it, may be the same as the right-hand side of (23) for small values of x_s, t, we must choose this numerical factor equal to $i\pi$. Hence expression (49) with u, v and w equal to $i\pi$ times their values in (29) must give the matrix elements

$(x't' \mid R_F \mid x''t'')$ with $t > 0$. The elements with $t < 0$ are given by (49) with u, v and w equal to $-i\pi$ times their values in (29).

The indeterminacy which we have in g will not lead to any indeterminacy in $(x't' \mid R_F \mid x''t'')$, since a variation in g causes v to change by a term containing $t^2 - r^2$ as a factor and such a term multiplied into $\delta(t^2 - r^2)$ will vanish. Thus R_F is completely fixed.

4. *Conclusion.*

From the foregoing work we see that the following results must hold, at least to the accuracy of the Hartree method of approximation:

(i) One can give a precise meaning to the distribution of electrons in which every state is occupied. This distribution may be defined as that described by the density matrix R_F given by (49), this matrix being completely fixed for any given field.

(ii) One can give a precise meaning to a distribution of electrons in which nearly all (i.e. all but a finite number, or all but a finite number per unit volume) of the negative-energy states are occupied and nearly all of the positive-energy ones unoccupied. Such a distribution may be defined as one described by a density matrix $R = \frac{1}{2}(R_F + R_1)$, where R_1 is of the form (48). This definition is permissible because the only possible variations in R_1, namely those due to g not being completely defined, are free from singularity and thus correspond to finite changes, or finite changes per unit volume, in the electron distribution. Our method does not give any precise meaning to which negative-energy states are unoccupied or which positive-energy ones are occupied. It is sufficiently definite, though, to take as the basis of the theory of the position the assumption that only distributions described by $R = \frac{1}{2}(R_F + R_1)$ with R_1 of the form (48) occur in nature.

(iii) A distribution R such as occurs in nature according to the above assumption can be divided naturally into two parts

$$R = R_a + R_b,$$

where R_a contains all the singularities and is also completely fixed for any given field, so that any alteration one may make in the distribution of electrons and positrons will correspond to an alteration in R_b but to none in R_a. We get this division into two parts by putting the term containing g into R_b and all the other terms into R_a. Thus

$$R_b = g/4i\hbar.$$

It is easily seen that R_b is relativistically invariant and gauge invariant, and it may be verified after some calculation that R_b is Hermitian and that the electric density and current density

corresponding to it satisfy the conservation law (9). It therefore appears reasonable to make the assumption that *the electric and current densities corresponding to R_b are those which are physically present, arising from the distribution of electrons and positrons.* In this way we can remove the infinities mentioned at the end of § 1.

The present paper is incomplete in that the effect of the exclusion principle, equation (2) or (6), on R_b has not been investigated. Further work that remains to be done is to examine the physical consequences of the foregoing assumption and to see whether it leads to any phenomena of the nature of a polarization of a vacuum by an electromagnetic field.