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Quantum Electrodynamics. II. Vacuum Polarization and Self-Energy

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The covariant formulation of quantum electrodynamics, developed in a previous paper, is here applied to two elementary problems—the polarization of the vacuum and the self-energies of the electron and photon. In the first section the vacuum of the non-interacting electromagnetic and matter fields is covariantly defined as that state for which the eigenvalue of an arbitrary time-like component of the energy-momentum four-vector is an absolute minimum. It is remarked that this definition must be compatible with the requirement that the vacuum expectation values of a physical quantity in various coordinate systems should be, not only covariantly related, but identical, since the vacuum has a significance that is independent of the coordinate system. In order to construct a suitable characterization of the vacuum state vector, a covariant decomposition of the field operators into positive and negative frequency components is introduced, and the properties of these associated fields developed. It is shown that the state vector for the electromagnetic vacuum is annihilated by the positive frequency part of the transverse four-vector potential, while that for the matter vacuum is annihilated by the positive frequency part of the Dirac spinor and of its charge conjugate. These defining properties of the vacuum state vector are employed in the calculation of the vacuum expectation values of quadratic field quantities, specifically the energy-momentum tensors of the independent electromagnetic and matter fields, and the current four-vector. It is inferred that the electromagnetic energy-momentum tensor, and the current vector must vanish in the vacuum, while the matter field energy-momentum tensor vanishes in the vacuum only by the addition of a suitable multiple of the unit tensor. The second section treats the induction of a current in the vacuum by an external electromagnetic field. It is supposed that the latter does not produce actual electron-positron pairs; that is, we consider only the phenomenon of virtual pair creation. This restriction is introduced by requiring that the establishment and subsequent removal of the external field produce no net change in state for the matter field. It is demonstrated, in a general manner, that the induced current at a given space-time point involves the external current in the vicinity of that point, and not the electromagnetic potentials. This gauge invariant result shows that a light wave, propagating at remote distances from its source, induces no current in the vacuum and is therefore undisturbed in its passage through space. The absence of a light quantum self-energy effect is thus indicated. The current induced at a point consists, more precisely, of two parts: a logarithmically divergent multiple of the external current at that point, which produces an unobservable renormalization of charge, and a more involved finite contribution, which is the physically significant induced current. The latter agrees with the results of previous investigations. The modification of the matter field properties arising from interaction with the vacuum fluctuations of the electromagnetic field is considered in the third section. The analysis is carried out with two alternative formulations, one employing the complete electromagnetic potential together with a supplementary condition, the other using the transverse potential, with the variables of the supplementary condition eliminated. It is noted that no real processes are produced by the first order coupling between the fields. Accordingly, alternative equations of motion for the state vector are constructed, from which the first order interaction term has been eliminated and replaced by the second order coupling which it generates. The latter includes the self action of individual particles and light quanta, the interaction of different particles, and a coupling between particles and light quanta which produces such effects as Compton scattering and two quantum pair annihilation. It is concluded from a comparison of the alternative procedures that, for the treatment of virtual light quantum processes, the separate consideration of longitudinal and transverse fields is an inadvisable complication. The light quantum self-energy term is shown to vanish, while that for a particle has the anticipated form for a change in proper mass, although the latter is logarithmically divergent, in agreement with previous calculations. To confirm the identification of the self-energy effect with a change in proper mass, it is shown that the result of removing this term from the state vector equation of motion is to alter the matter field equations of motion in the expected manner. It is verified, finally, that the energy and momentum modifications produced by self-interaction effects are entirely accounted for by the addition of the electromagnetic proper mass to the mechanical proper mass—an unobservable mass renormalization. An appendix is devoted to the construction of several invariant functions associated with the electromagnetic and matter fields.

THE first article of this series[1] was concerned with a formulation of quantum electrodynamics that has the following essential features—explicit covariance with respect to Lorentz transformations, and a natural division between the properties of independent fields and the effects of field interactions. As the simplest

[1] Quantum electrodynamics. I. A covariant formulation, Phys. Rev. **74**, 1439 (1948), hereinafter referred to as I. References to equations in the work will be written in the typical form (I, 2.3).

example of the latter, we consider, in this second paper, the phenomena of vacuum polarization and the self energies of photon and electron, which arise from the coupling between the matter and electromagnetic fields and their vacuum fluctuations. It will first be necessary to construct a suitable covariant definition for the vacuum of the independent matter and electromagnetic fields.

1. DEFINITION OF THE VACUUM

In order to define the vacuum of the electromagnetic fields, it is convenient to introduce two auxiliary four-vector fields that obey the same differential equation as $A_\mu(x)$

$$A_\mu^{(+)}(x)=\frac{1}{2\pi i}\int_{C_+} A_\mu(x-\epsilon\tau)\frac{d\tau}{\tau}, \quad (1.1a)$$

$$A_\mu^{(-)}(x)=\frac{1}{2\pi i}\int_{C_+} A_\mu(x+\epsilon\tau)\frac{d\tau}{\tau}, \quad (1.1b)$$

in which the contour C_+ is extended from $-\infty$ to $+\infty$, deformed below the singularity at $\tau=0$. It is intended that ϵ_μ be a time-like vector, $\epsilon_\mu^2<0$, with $\epsilon_0=(1/i)\epsilon_4>0$, which are covariant requirements. The fields $A_\mu^{(+)}(x)$ and $A_\mu^{(-)}(x)$ are then independent of the special choice of ϵ_μ. A simple connection between the three fields is obtained by rewriting (1.1b) as

$$A_\mu^{(-)}(x)=\frac{1}{2\pi i}\int_{C_-} A_\mu(x-\epsilon\tau)\frac{d\tau}{\tau}, \quad (1.2)$$

where C_- is extended from $+\infty$ to $-\infty$, deformed above the singularity at $\tau=0$. Evidently the sum of the contours C_+ and C_- is a closed contour drawn about the singularity at the origin, whence

$$A_\mu^{(+)}(x)+A_\mu^{(-)}(x)=\frac{1}{2\pi i}\oint A_\mu(x-\epsilon\tau)\frac{d\tau}{\tau}=A_\mu(x). \quad (1.3)$$

On choosing the contours C_+ and C_- to coincide with the real axis, save for appropriate semicircles of negligible radius drawn about the origin, one obtains from (1.1a) and (1.2) the forms

$$\begin{aligned}A_\mu^{(+)}(x)&=\tfrac{1}{2}[A_\mu(x)-iA_\mu^{(1)}(x)],\\ A_\mu^{(-)}(x)&=\tfrac{1}{2}[A_\mu(x)+iA_\mu^{(1)}(x)],\end{aligned} \quad (1.4)$$

where

$$\begin{aligned}A_\mu^{(1)}(x)&=i[A_\mu^{(+)}(x)-A_\mu^{(-)}(x)],\\ &=\frac{1}{\pi}P\int_{-\infty}^{\infty} A_\mu(x-\epsilon\tau)\frac{d\tau}{\tau},\end{aligned} \quad (1.5)$$

and P symbolizes the principal part of the integral. It will be noted that $A_1^{(1)}$, $A_2^{(1)}$, $A_3^{(1)}$, and $A_0^{(1)}=(1/i)A_4^{(1)}$ are Hermitian operators, in view of the Hermitian character of the corresponding components of $A_\mu(x)$. Accordingly, $A_\mu^{(-)}$ is the Hermitian conjugate of $A_\mu^{(+)}$, where $\mu=0, 1, 2, 3$.

The significance of the associated fields $A_\mu^{(+)}(x)$, $A_\mu^{(-)}(x)$, and $A_\mu^{(1)}(x)$ can best be appreciated in terms of a Fourier representation of $A_\mu(x)$,

$$A_\mu(x)=\int A_\mu(k)\delta(k_\lambda^2)\exp(ik_\mu x_\mu)(dk), \quad (1.6)$$

where $(dk)=dk_0dk_1dk_2dk_3$ is the four-dimensional volume element in wave number, or k space. The delta function, $\delta(k_\lambda^2)$, ensures that the individual terms of the expansion obey the wave Eq. (I, 2.11), leaving the Fourier amplitudes, $A_\mu(k)$, quite arbitrary. According to the definition (1.1a) and (1.1b),

$$\begin{aligned}A_\mu^{(+)}(x)&=\int A_\mu(k)\delta(k_\lambda^2)\exp(ik_\mu x_\mu)(dk)\frac{1}{2\pi i}\times\int_{C_+}\exp(-ik_\lambda\epsilon_\lambda\tau)\frac{d\tau}{\tau},\\ A_\mu^{(-)}(x)&=\int A_\mu(k)\delta(k_\lambda^2)\exp(ik_\mu x_\mu)(dk)\frac{1}{2\pi i}\times\int_{C_+}\exp(ik_\lambda\epsilon_\lambda\tau)\frac{d\tau}{\tau}.\end{aligned} \quad (1.7)$$

But

$$\frac{1}{2\pi i}\int_{C_+}\exp(-ik_\lambda\epsilon_\lambda\tau)\frac{d\tau}{\tau}=\begin{cases}1, & -k_\lambda\epsilon_\lambda>0\\ 0, & -k_\lambda\epsilon_\lambda<0,\end{cases} \quad (1.8)$$

whence

$$A_\mu^{(+)}(x)=\int_{-k_\lambda\epsilon_\lambda>0} A_\mu(k)\delta(k_\lambda^2)\times\exp(ik_\mu x_\mu)(dk), \quad (1.9)$$

and

$$A_\mu^{(-)}(x) = \int_{-k_\lambda\epsilon_\lambda<0} A_\mu(k)\delta(k_\lambda^2) \times\exp(ik_\mu x_\mu)(dk). \quad (1.10)$$

The two domains in wave number space characterized by the sign of $-k_\mu\epsilon_\mu$ are actually independent of ϵ_μ, provided the latter is a time-like vector with a definite sign for ϵ_0. It is sufficient to observe that

$$-k_\mu\epsilon_\mu = k_0\epsilon_0\left(1-\frac{\mathbf{k}\cdot\boldsymbol{\varepsilon}}{k_0\epsilon_0}\right), \quad (1.11)$$

and

$$\left|\frac{\mathbf{k}\cdot\boldsymbol{\varepsilon}}{k_0\epsilon_0}\right| < \frac{|\mathbf{k}|}{|k_0|}\frac{|\boldsymbol{\varepsilon}|}{|\epsilon_0|} < 1, \quad (1.12)$$

in which the last inequality presupposes that ϵ_μ is time-like and k_μ either a null vector or a time-like vector. Accordingly, the sign of $-k_\mu\epsilon_\mu$ is determined by that of $k_0\epsilon_0$, and if ϵ_0 is restricted to be positive, itself an invariant requirement, it is inferred that $-k_\mu\epsilon_\mu$ possesses the algebraic sign of k_0. It is now clear that if $A_\mu(x)$ is constructed as an arbitrary superposition of plane waves,

$$\exp(ik_\mu x_\mu) = \exp[i(\mathbf{k}\cdot\mathbf{r}-k_0x_0)],$$

the functions $A_\mu^{(+)}(x)$ and $A_\mu^{(-)}(x)$ consist of the positive and negative frequency parts, respectively, which is an invariant decomposition. The function $A_\mu^{(1)}(x)$ contains both the positive and negative frequency parts of $A_\mu(x)$, but with the factors i and $-i$, respectively.

The commutation properties of the auxiliary fields with $A_\mu(x)$ are easily constructed. According to the definitions and Eq. (I, 2.28),

$$\begin{aligned}
[A_\mu^{(+)}(x), A_\nu(x')] &= i\hbar c\delta_{\mu\nu}D^{(+)}(x-x'),\\
[A_\mu^{(-)}(x), A_\nu(x')] &= i\hbar c\delta_{\mu\nu}D^{(-)}(x-x'), \quad (1.13)\\
[A_\mu^{(1)}(x), A_\nu(x')] &= i\hbar c\delta_{\mu\nu}D^{(1)}(x-x'),
\end{aligned}$$

where

$$\begin{aligned}
D^{(+)}(x) &= \frac{1}{2\pi i}\int_{C_+} D(x-\epsilon\tau)\frac{d\tau}{\tau},\\
D^{(-)}(x) &= \frac{1}{2\pi i}\int_{C_-} D(x-\epsilon\tau)\frac{d\tau}{\tau}, \quad (1.14)\\
D^{(1)}(x) &= \frac{1}{\pi}P\int_{-\infty}^{\infty} D(x-\epsilon\tau)\frac{d\tau}{\tau}.
\end{aligned}$$

Relations analogous to (1.3) and (1.4) are valid, of course, for the various D functions:

$$\begin{aligned}
D^{(+)}(x)+D^{(-)}(x) &= D(x),\\
D^{(+)}(x) &= \tfrac{1}{2}[D(x)-iD^{(1)}(x)], \quad (1.15)\\
D^{(-)}(x) &= \tfrac{1}{2}[D(x)+iD^{(1)}(x)].
\end{aligned}$$

In addition, $D^{(1)}(x)$ shares the reality property of $D(x)$, and $D^{(-)}(x)$ is the complex conjugate of $D^{(+)}(x)$. Further, the odd nature of $D(x)$, as a function of the coordinates, implies that $D^{(1)}(x)$ is an even function

$$D^{(1)}(-x) = D^{(1)}(x), \quad (1.16)$$

which has as a consequence that

$$D^{(-)}(x) = -D^{(+)}(-x). \quad (1.17)$$

Finally, the validity of the differential equations

$$\square^2 D^{(+)}(x) = \square^2 D^{(-)}(x) = \square^2 D^{(1)}(x) = 0 \quad (1.18)$$

will be evident.

The essential commutation properties of the auxiliary fields are contained in the statements

$$[A_\mu^{(+)}(x), A_\nu^{(+)}(x')] = [A_\mu^{(-)}(x), A_\nu^{(-)}(x')] = 0. \quad (1.19)$$

It follows by direct calculation that

$$\begin{aligned}
[A_\mu^{(+)}(x), A_\nu^{(+)}(x')] &= i\hbar c\delta_{\mu\nu}\frac{1}{(2\pi i)^2}\int_{C_+} D(x-x'-\epsilon(\tau-\tau'))\frac{d\tau}{\tau}\frac{d\tau'}{\tau'}\\
&= i\hbar c\delta_{\mu\nu}\frac{1}{2\pi}\int_{-\infty}^{\infty} d\alpha e^{i\alpha\lambda}D(x-x'-\epsilon\lambda)\frac{1}{2\pi i}\\
&\quad\times\int_{C_+} e^{i\alpha\tau}\frac{d\tau}{\tau}\frac{1}{2\pi i}\int_{C_+} e^{-i\alpha\tau'}\frac{d\tau'}{\tau'} = 0, \quad (1.20)
\end{aligned}$$

in which the decisive steps are the introduction of a Fourier integral representation for the τ dependence of $D(x-\epsilon\tau)$, and the observation (see Eq. (1.8)) that the two resulting contour integrals are never simultaneously different from zero (except for the isolated point $\alpha=0$). Alternatively, we can remark that the positive frequency character of $A_\mu^{(+)}(x)$ is incompatible with the physical requirement that the commutator involve only $x_\mu - x_\mu'$, the interval between the two points x and x', unless the commutator vanishes. The proof for $A_\mu^{(-)}$ is identical. The commutator of $A_\mu^{(+)}(x)$ with $A_\nu^{(-)}(x')$ can also be directly evaluated along the lines of (1.20), but it is sufficient to combine (1.19) with (1.13) to obtain

$$[A_\mu^{(+)}(x), A_\nu^{(-)}(x')] = i\hbar c\delta_{\mu\nu}D^{(+)}(x-x'), \quad (1.21)$$

or

$$[A_\mu^{(-)}(x), A_\nu^{(+)}(x')] = i\hbar c\delta_{\mu\nu}D^{(-)}(x-x'). \quad (1.22)$$

It should also be noted that, in view of the identity

$$\begin{aligned}[A_\mu^{(1)}(x), A_\nu^{(1)}(x')] - [A_\mu(x), A_\nu(x')] \\ = -2[A_\mu^{(+)}(x), A_\nu^{(+)}(x')] \\ -2[A_\mu^{(-)}(x), A_\nu^{(-)}(x')], \quad (1.23)\end{aligned}$$

the commutation relations (1.19) imply that

$$\begin{aligned}[A_\mu^{(1)}(x), A_\nu^{(1)}(x')] &= [A_\mu(x), A_\nu(x')] \\ &= i\hbar c\delta_{\mu\nu}D(x-x').\end{aligned} \quad (1.24)$$

In a manner quite analogous to that presented for $A_\mu(x)$, various associated functions can be defined for $\mathfrak{A}_\mu(x)$, $\Lambda(x)$, and $\Lambda'(x)$. Without entering into repetitious detail, let us merely record the definitions

$$\begin{aligned}\mathfrak{A}_\mu^{(+)}(x) &= \tfrac{1}{2}[\mathfrak{A}_\mu(x) - i\mathfrak{A}_\mu^{(1)}(x)], \\ \mathfrak{A}_\mu^{(-)}(x) &= \tfrac{1}{2}[\mathfrak{A}_\mu(x) + i\mathfrak{A}_\mu^{(1)}(x)],\end{aligned} \quad (1.25)$$

and the commutation relations

$$[\mathfrak{A}_\mu^{(+)}(x), \mathfrak{A}_\nu^{(+)}(x')] = [\mathfrak{A}_\mu^{(-)}(x), \mathfrak{A}_\nu^{(-)}(x')] = 0, \quad (1.26)$$

$$\begin{aligned}[\mathfrak{A}_\mu^{(1)}(x), \mathfrak{A}_\nu(x')] &= i\hbar c\delta_{\mu\nu}D^{(1)}(x-x') \\ &- i\hbar c\left[\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu} + \left(n_\mu\frac{\partial}{\partial x_\nu} + n_\nu\frac{\partial}{\partial x_\mu}\right)n_\lambda\frac{\partial}{\partial x_\lambda}\right] \\ &\times \mathfrak{D}^{(1)}(x-x'). \quad (1.27)\end{aligned}$$

We shall consider the vacuum of the isolated electromagnetic field to be that state for which the eigenvalue of the energy, or better, an arbitrary time-like component of the energy-momentum four-vector, is an absolute minimum. This definition must also be compatible with the evident requirement that the vacuum expectation values of a physical quantity in various coordinate systems should be, not only covariantly related, but identical; the properties of the vacuum are independent of the coordinate system. In order to utilize the minimum energy definition, we apply (I, 1.38) to the positive frequency part of the physically significant electromagnetic field vector $\mathfrak{A}_\mu(x)$,

$$\mathfrak{A}_\mu^{(+)}(x)P_\nu - P_\nu\mathfrak{A}_\mu^{(+)}(x) = \frac{\hbar}{i}\frac{\partial}{\partial x_\nu}\mathfrak{A}_\mu^{(+)}(x). \quad (1.28)$$

On considering a particular Fourier component of $\mathfrak{A}_\mu^{(+)}(x)$, the latter equation becomes

$$\mathfrak{A}_\mu^{(+)}(k)P_\nu - P_\nu\mathfrak{A}_\mu^{(+)}(k) = \hbar k_\nu\mathfrak{A}_\mu^{(+)}(k), \quad (1.29)$$

which further reduces to

$$\mathfrak{A}_\mu^{(+)}(k)W - W\mathfrak{A}_\mu^{(+)}(k) = \hbar\omega\mathfrak{A}_\mu^{(+)}(k) \quad (1.30)$$

on multiplication with a unit time-like vector ϵ_ν such that $\epsilon_0 > 0$.

Here

$$W = -\epsilon_\nu P_\nu c, \quad \omega = -\epsilon_\nu k_\nu c \quad (1.31)$$

represent invariant expressions for energy and frequency in an arbitrary coordinate system specified by ϵ_ν. We may now apply both sides of (1.30) to the state vector Ψ_0 representing the vacuum of the electromagnetic field and obtain

$$W[\mathfrak{A}_\mu^{(+)}(k)\Psi_0] = (W_0 - \hbar\omega)[\mathfrak{A}_\mu^{(+)}(k)\Psi_0], \quad (1.32)$$

where W_0 is the eigenvalue of W in the state described by Ψ_0. This result implies that in the state described by $\mathfrak{A}_\mu^{(+)}(k)\Psi_0$, W has the eigenvalue $W_0 - \hbar\omega$. Inasmuch as the defining property of $\mathfrak{A}_\mu^{(+)}(k)$ guarantees that ω is positive, we are confronted with a state of lower energy than that in the vacuum. This contradiction can be resolved only if

$$\mathfrak{A}_\mu^{(+)}(k)\Psi_0 = 0, \quad (1.33)$$

which serves to specify Ψ_0. Since (1.33) is valid for all k, we may write

$$\mathfrak{A}_\mu^{(+)}(x)\Psi_0 = 0, \quad (1.34)$$

which is self-consistent, in view of the commutation properties of $\mathfrak{A}_\mu^{(+)}$.

The definition of the vacuum thus obtained can be used to evaluate vacuum expectation values of quadratic field quantities, of which the basic form is

$$\{\mathfrak{A}_\mu(x),\ \mathfrak{A}_\nu(x')\} = \mathfrak{A}_\mu(x)\mathfrak{A}_\nu(x') + \mathfrak{A}_\nu(x')\mathfrak{A}_\mu(x).$$

Now

$$\begin{aligned}\{\mathfrak{A}_\mu(x),\ \mathfrak{A}_\nu(x')\} &= (2\mathfrak{A}_\mu^{(-)}(x) - i\mathfrak{A}_\mu^{(1)}(x))\mathfrak{A}_\nu(x') \\ &\qquad + \mathfrak{A}_\nu(x')(2\mathfrak{A}_\mu^{(+)}(x) + i\mathfrak{A}_\mu^{(1)}(x)) \\ &= -i[\mathfrak{A}_\mu^{(1)}(x),\ \mathfrak{A}_\nu(x')] \\ &\qquad + 2(\mathfrak{A}_\mu^{(-)}(x)\mathfrak{A}_\nu(x') + \mathfrak{A}_\nu(x')\mathfrak{A}_\mu^{(+)}(x)), \qquad (1.35)\end{aligned}$$

where the first term is a known commutator and the second has a vanishing vacuum expectation value. Thus

$$\begin{aligned}&\langle \mathfrak{A}_\mu^{(-)}(x)\mathfrak{A}_\nu(x') + \mathfrak{A}_\nu(x')\mathfrak{A}_\mu^{(+)}(x)\rangle_0 \\ &= (\Psi_0,\ (\mathfrak{A}_\mu^{(-)}(x)\mathfrak{A}_\nu(x') + \mathfrak{A}_\nu(x')\mathfrak{A}_\nu^{(+)}(x))\Psi_0) \\ &= \pm(\mathfrak{A}_\mu^{(+)}(x)\Psi_0,\ \mathfrak{A}_\nu(x')\Psi_0) \\ &\qquad + (\Psi_0,\ \mathfrak{A}_\nu(x')\mathfrak{A}_\mu^{(+)}(x)\Psi_0) = 0, \qquad (1.36)\end{aligned}$$

which involves the fact that, to within a minus sign for $\mu=4$, the Hermitian conjugate of $\mathfrak{A}_\mu^{(-)}$ is $\mathfrak{A}_\mu^{(+)}$. Hence,

$$\begin{aligned}\langle\{\mathfrak{A}_\mu(x),\ \mathfrak{A}_\nu(x')\}\rangle_0 &= \hbar c\delta_{\mu\nu}D^{(1)}(x-x') \\ &\quad - \hbar c\left[\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu} + \left(n_\mu\frac{\partial}{\partial x_\nu} + n_\nu\frac{\partial}{\partial x_\mu}\right)n_\lambda\frac{\partial}{\partial x_\lambda}\right] \\ &\qquad \times \mathfrak{D}^{(1)}(x-x'). \qquad (1.37)\end{aligned}$$

From this result, one can compute that

$$\begin{aligned}&\langle\{\mathfrak{F}_{\mu\lambda}(x),\ \mathfrak{F}_{\nu\sigma}(x')\}\rangle_0 \\ &= \hbar c\left(\delta_{\lambda\nu}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\sigma} + \delta_{\mu\sigma}\frac{\partial}{\partial x_\lambda}\frac{\partial}{\partial x_\nu} - \delta_{\lambda\sigma}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\right. \\ &\qquad \left. - \delta_{\mu\nu}\frac{\partial}{\partial x_\lambda}\frac{\partial}{\partial x_\sigma}\right)D^{(1)}(x-x'), \qquad (1.38)\end{aligned}$$

of which two successive specializations are

$$\begin{aligned}&\langle\{\mathfrak{F}_{\mu\lambda}(x),\ \mathfrak{F}_{\nu\lambda}(x')\}\rangle_0 \\ &= -\hbar c\left(2\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu} + \delta_{\mu\nu}\square^2\right)D^{(1)}(x-x') \\ &= -2\hbar c\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}D^{(1)}(x-x'), \qquad (1.39)\end{aligned}$$

and

$$\begin{aligned}\langle\{\mathfrak{F}_{\lambda\sigma}(x),\ \mathfrak{F}_{\lambda\sigma}(x')\}\rangle_0 &= -2\hbar c\square^2 D^{(1)}(x-x') \\ &= 0. \qquad (1.40)\end{aligned}$$

As an elementary application of these vacuum expectation values, one may consider the computation of the vacuum average value of the mechanical quantities comprised in the electromagnetic energy-momentum tensor

$$\begin{aligned}\Theta_{\mu\nu} &= \tfrac{1}{2}\{\mathfrak{F}_{\mu\lambda}(x),\ \mathfrak{F}_{\nu\lambda}(x)\} \\ &\qquad - \tfrac{1}{8}\delta_{\mu\nu}\{\mathfrak{F}_{\lambda\sigma}(x),\ \mathfrak{F}_{\lambda\sigma}(x)\}, \qquad (1.41)\end{aligned}$$

which has the important property of being trace-less,

$$\Theta_{\mu\mu} = 0. \qquad (1.42)$$

Evidently,

$$\langle\Theta_{\mu\nu}\rangle_0 = -\hbar c\frac{\partial}{\partial\xi_\mu}\frac{\partial}{\partial\xi_\nu}D^{(1)}(\xi)\bigg]_{\xi=0}, \qquad (1.43)$$

which is indeterminate in view of the singularity of $D^{(1)}(\xi)$ as $\xi_\mu^2 \to 0$. However, the form of (1.43) can be inferred from quite general requirements. Since $D^{(1)}(\xi)$ is a function only of ξ_μ^2, the tensor resulting from the indicated process can only be a multiple of $\delta_{\mu\nu}$,

$$\frac{\partial}{\partial\xi_\mu}\frac{\partial}{\partial\xi_\nu}D^{(1)}(\xi)\bigg]_{\xi=0} = K\delta_{\mu\nu}, \qquad (1.44)$$

and, on placing $\mu=\nu$, with the implied summation, we learn that

$$4K = \square^2 D^{(1)}(\xi)\bigg]_{\xi=0} = 0, \qquad (1.45)$$

whence

$$\langle\Theta_{\mu\nu}\rangle_0 = 0. \qquad (1.46)$$

This, indeed, is the only result compatible with the requirement that the properties of the vacuum be independent of the coordinate system. The values ascribed to the symmetrical tensor $\langle\Theta_{\mu\nu}\rangle_0$ can be identical in all coordinate systems only if the tensor is a multiple of $\delta_{\mu\nu}$. If, in addition, it is restricted to be trace-less, the tensor must vanish. Thus, a non-vanishing electromagnetic vacuum fluctuation energy is incompatible with relativistic requirements.

The process of defining the vacuum of the matter field follows the pattern that has been presented for the electromagnetic field, with appropriate modifications. The spinors

$$\psi^{(+)}(x)=\frac{1}{2\pi i}\int_{C_+}\psi(x-\epsilon\tau)\frac{d\tau}{\tau} \tag{1.47}$$

and

$$\psi^{(-)}(x)=\frac{1}{2\pi i}\int_{C_+}\psi(x+\epsilon\tau)\frac{d\tau}{\tau} \tag{1.48}$$

are the positive and negative frequency (or energy) parts of $\psi(x)$. They may be written

$$\begin{aligned}\psi^{(+)}(x)&=\tfrac{1}{2}[\psi(x)-i\psi^{(1)}(x)],\\ \psi^{(-)}(x)&=\tfrac{1}{2}[\psi(x)+i\psi^{(1)}(x)],\end{aligned} \tag{1.49}$$

where

$$\psi^{(1)}(x)=\frac{1}{\pi}P\int_{-\infty}^{\infty}\psi(x-\epsilon\tau)\frac{d\tau}{\tau}. \tag{1.50}$$

The same operations can be applied to the adjoint spinor $\bar{\psi}(x)$. In particular,

$$\bar{\psi}^{(1)}(x)=\frac{1}{\pi}P\int_{-\infty}^{\infty}\bar{\psi}(x-\epsilon\tau)\frac{d\tau}{\tau}, \tag{1.51}$$

which makes it evident that

$$\bar{\psi}^{(1)}(x)=\overline{\psi^{(1)}}(x). \tag{1.52}$$

This, in turn, implies the relations

$$\bar{\psi}^{(+)}(x)=\overline{\psi^{(-)}}(x),\quad \bar{\psi}^{(-)}(x)=\overline{\psi^{(+)}}(x), \tag{1.53}$$

replacing the simpler reality properties of the functions associated with the electromagnetic four-vector potential. The decomposition of the charge conjugate spinors follows directly from the definitions (I, 1.3). Thus,

$$\psi'^{(+)}=C\bar{\psi}^{(+)}=C\overline{\psi^{(-)}},\quad \psi'^{(-)}=C\bar{\psi}^{(-)}=C\overline{\psi^{(+)}}, \tag{1.54}$$

and

$$\begin{aligned}\bar{\psi}'^{(+)}&=\overline{\psi'^{(-)}}=C^{-1}\psi^{(+)},\\ \bar{\psi}'^{(-)}&=\overline{\psi'^{(+)}}=C^{-1}\psi^{(-)}.\end{aligned} \tag{1.55}$$

The commutation relations of ψ with the various associated fields can be constructed immediately. In particular,

$$\begin{aligned}\{\psi_\alpha^{(1)}(x),\bar{\psi}_\beta(x')\}&=-\{\psi_\alpha(x),\overline{\psi_\beta^{(1)}}(x')\}\\ &=\frac{1}{i}S_{\alpha\beta}^{(1)}(x-x')\\ &=\frac{1}{i}\left(\gamma_\mu\frac{\partial}{\partial x_\mu}-\kappa_0\right)_{\alpha\beta}\Delta^{(1)}(x-x')\end{aligned} \tag{1.56}$$

where

$$\Delta^{(1)}(x)=\frac{1}{\pi}P\int_{-\infty}^{\infty}\Delta(x-\epsilon\tau)\frac{d\tau}{\tau} \tag{1.57}$$

is an even function of x obeying the differential equation

$$(\Box^2-\kappa_0^2)\Delta^{(1)}(x)=0. \tag{1.58}$$

The explicit construction of this and the other functions that have been introduced is performed in the Appendix. The elementary arguments used to prove (1.19) will also serve to demonstrate that

$$\begin{aligned}\{\psi_\alpha^{(+)}(x),\bar{\psi}_\beta^{(+)}(x')\}&=\{\psi_\alpha^{(+)}(x),\overline{\psi_\beta^{(-)}}(x')\}=0,\\ \{\psi_\alpha^{(-)}(x),\bar{\psi}_\beta^{(-)}(x')\}&=\{\psi_\alpha^{(-)}(x),\overline{\psi_\beta^{(+)}}(x')\}=0,\end{aligned} \tag{1.59}$$

whence

$$\begin{aligned}\{\psi_\alpha^{(+)}(x),\overline{\psi_\beta^{(+)}}(x')\}&=\frac{1}{i}S_{\alpha\beta}^{(+)}(x-x'),\\ \{\psi_\alpha^{(-)}(x),\overline{\psi_\beta^{(-)}}(x')\}&=\frac{1}{i}S_{\alpha\beta}^{(-)}(x-x').\end{aligned} \tag{1.60}$$

It can also be shown, analogously to (1.23), that

$$\begin{aligned}\{\psi_\alpha^{(1)}(x),\overline{\psi_\beta^{(1)}}(x')\}&=\{\psi_\alpha(x),\bar{\psi}_\beta(x')\}\\ &=\frac{1}{i}S_{\alpha\beta}(x-x').\end{aligned} \tag{1.61}$$

All such commutation relations are invariant with respect to charge conjugation. As an example,

$$\begin{aligned}\{\psi_\alpha'^{(1)}(x),\bar{\psi}_\beta'(x')\}&\\ &=\{(C\bar{\psi}^{(1)}(x))_\alpha,(C^{-1}\psi(x'))_\beta\}\\ &=-C_{\alpha\gamma}\{\psi_\delta(x'),\bar{\psi}_\gamma^{(1)}(x)\}C_{\delta\beta}^{-1}\\ &=\frac{1}{i}\left(C\gamma_\mu^TC^{-1}\frac{\partial}{\partial x_\mu'}-\kappa_0\right)_{\alpha\beta}\Delta^{(1)}(x'-x)\\ &=\frac{1}{i}\left(\gamma_\mu\frac{\partial}{\partial x_\mu}-\kappa_0\right)_{\alpha\beta}\Delta^{(1)}(x-x')\\ &=\frac{1}{i}S_{\alpha\beta}^{(1)}(x-x').\end{aligned} \tag{1.62}$$

The characterization of the matter field vacuum as the state of minimum energy can be exploited, in complete analogy with the procedure for the electromagnetic field, to yield the following defining equations for the vacuum state vector Ψ_0

$$\psi^{(+)}(x)\Psi_0=0, \tag{1.63}$$

and

$$\bar{\psi}^{(+)}(x)\Psi_0=\overline{\psi^{(-)}}(x)\Psi_0=0. \tag{1.64}$$

The latter equation can also be written as the charge conjugate form of (1.63)

$$\psi'^{(+)}(x)\Psi_0=0. \tag{1.65}$$

In order to evaluate the vacuum expectation value of the typical bilinear expression

$$[\psi_\alpha(x), \bar{\psi}_\beta(x')]=\psi_\alpha(x)\bar{\psi}_\beta(x')-\bar{\psi}_\beta(x')\psi_\alpha(x),$$

we write

$$\begin{aligned}[\psi_\alpha(x), \bar{\psi}_\beta(x')] &= (2\psi_\alpha^{(-)}(x)-i\psi_\alpha^{(1)}(x))\bar{\psi}_\beta(x') \\ &\quad -\bar{\psi}_\beta(x')(2\psi_\alpha^{(+)}(x)+i\psi_\alpha^{(1)}(x)) \\ &= -i\{\psi_\alpha^{(1)}(x), \bar{\psi}_\beta(x')\} \\ &\quad +2(\psi_\alpha^{(-)}(x)\bar{\psi}_\beta(x')-\bar{\psi}_\beta(x')\psi_\alpha^{(+)}(x)),\end{aligned} \tag{1.66}$$

and observe that the vacuum expectation value of the second term is zero.

$$\begin{aligned}(\Psi_0, (\psi_\alpha^{(-)}(x)\bar{\psi}_\beta(x')-\bar{\psi}_\beta(x')\psi_\alpha^{(+)}(x))\Psi_0) &= (\psi_\alpha^{(-)\dagger}(x)\Psi_0, \bar{\psi}_\beta(x')\Psi_0) \\ &\quad -(\Psi_0, \bar{\psi}_\beta(x')\psi_\alpha^{(+)}(x)\Psi_0)=0,\end{aligned} \tag{1.67}$$

since the Hermitian conjugate and adjoint spinors are linearly related. Hence

$$\begin{aligned}\langle[\psi_\alpha(x), \bar{\psi}_\beta(x')]\rangle_0 &= \langle[\psi_\alpha'(x), \bar{\psi}_\beta'(x')]\rangle_0 \\ &= -S_{\alpha\beta}^{(1)}(x-x').\end{aligned} \tag{1.68}$$

We may apply this result to the evaluation of the expectation values of the current four-vector,

$$\begin{aligned}j_\mu &= -\frac{iec}{2}[\psi_\alpha(x), \bar{\psi}_\beta(x)](\gamma_\mu)_{\beta\alpha} \\ &= \frac{iec}{2}[\psi_\alpha'(x), \bar{\psi}_\beta'(x)](\gamma_\mu)_{\beta\alpha},\end{aligned} \tag{1.69}$$

and the symmetrical energy-momentum tensor of the matter field (see (I, 1.29)),

$$\begin{aligned}\Theta_{\mu\nu} = -\frac{\hbar c}{4}\bigg[&(\gamma_\mu)_{\beta\alpha}\frac{\partial}{\partial\xi_\nu}[\psi_\alpha(x+\tfrac{1}{2}\xi), \bar{\psi}_\beta(x-\tfrac{1}{2}\xi)] \\ &+(\gamma_\nu)_{\beta\alpha}\frac{\partial}{\partial\xi_\mu}[\psi_\alpha(x+\tfrac{1}{2}\xi), \bar{\psi}_\beta(x-\tfrac{1}{2}\xi)]\bigg]_{\xi=0},\end{aligned} \tag{1.70}$$

the trace of which is (see (I, 1.34))

$$\Theta_{\mu\mu}=m_0c^2\tfrac{1}{2}[\psi_\alpha(x), \bar{\psi}_\alpha(x)]. \tag{1.71}$$

On adding the two equivalent, charge conjugate, expressions for the current vector we find, according to (1.68), that

$$\begin{aligned}\langle j_\mu\rangle_0 = \frac{iec}{4}(\gamma_\mu)_{\beta\alpha}(\langle[\psi_\alpha'(x), \bar{\psi}_\beta'(x)]\rangle_0 \\ -\langle[\psi_\alpha(x), \bar{\psi}_\beta(x)]\rangle_0)=0,\end{aligned} \tag{1.72}$$

which is a simple expression of the charge symmetry of the theory. Alternatively, direct calculation yields

$$\begin{aligned}\langle j_\mu\rangle_0 &= -\frac{iec}{2}\langle[\psi_\alpha(x), \bar{\psi}_\beta(x)]\rangle_0(\gamma_\mu)_{\beta\alpha} \\ &= \frac{iec}{2}TrS^{(1)}(\xi)\gamma_\mu\bigg]_{\xi=0} \\ &= 2iec\frac{\partial}{\partial\xi_\mu}\Delta^{(1)}(\xi)\bigg]_{\xi=0}=0,\end{aligned} \tag{1.73}$$

since $\Delta^{(1)}(\xi)$ is an even function. In (1.73) the symbol Tr indicates the trace, or diagonal sum, of the Dirac matrices. The following trace evaluations have been used:

$$Tr\gamma_\mu\gamma_\nu=4\delta_{\mu\nu}, \quad Tr\gamma_\mu=0, \tag{1.74}$$

the proofs of which involve only the anti-commutation properties of the γ_μ, and elementary theorems concerning traces. Thus,

$$Tr\gamma_\mu\gamma_\nu=Tr\tfrac{1}{2}(\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu)=4\delta_{\mu\nu}$$

and

$$Tr\gamma_\mu=Tr\tfrac{1}{2}\gamma_\mu(\gamma_5\gamma_5+\gamma_5\gamma_5)=Tr\tfrac{1}{2}(\gamma_\mu\gamma_5+\gamma_5\gamma_\mu)\gamma_5=0.$$

In the latter proof, $\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4$ forms the fifth Dirac matrix, completing the set obeying the anti-commutation relations (I, 1.1).

The vacuum expectation value of the energy-momentum tensor is

$$\langle\Theta_{\mu\nu}\rangle_0=\frac{\hbar c}{4}Tr\left(\gamma_\mu\frac{\partial}{\partial\xi_\nu}+\gamma_\nu\frac{\partial}{\partial\xi_\mu}\right)S^{(1)}(\xi)\Big]_{\xi=0}$$
$$=2\hbar c\frac{\partial}{\partial\xi_\mu}\frac{\partial}{\partial\xi_\nu}\Delta^{(1)}(\xi)\Big]_{\xi=0}. \tag{1.75}$$

The trace of $\langle\Theta_{\mu\nu}\rangle_0$ can be computed directly from (1.71),

$$\langle\Theta_{\mu\mu}\rangle_0=-m_0c^2\tfrac{1}{2}TrS^{(1)}(\xi)]_{\xi=0}$$
$$=2\hbar c\kappa_0^2\Delta^{(1)}(\xi)]_{\xi=0}, \tag{1.76}$$

which result also follows from (1.75). According to the general arguments presented in connection with the electromagnetic energy-momentum tensor, $\langle\Theta_{\mu\nu}\rangle_0$ must be a multiple of $\delta_{\mu\nu}$:

$$\langle\Theta_{\mu\nu}\rangle_0=K\delta_{\mu\nu}, \tag{1.77}$$

with

$$K=\tfrac{1}{4}\langle\Theta_{\mu\mu}\rangle_0=\tfrac{1}{2}\hbar c\kappa_0^2\Delta^{(1)}(0). \tag{1.78}$$

Unlike the electromagnetic field situation, the trace of the matter energy-momentum tensor does not vanish and, indeed, is divergent. There can be no objection, however, to altering the definition of the energy-momentum tensor by the addition of a suitable multiple of $\delta_{\mu\nu}$, which is so chosen that the vacuum expectation value of $\Theta_{\mu\nu}$ is zero.

2. THE POLARIZATION OF THE VACUUM

The first problem to which we turn our attention is the induction of a current in the matter field vacuum by an electromagnetic field—the polarization of the vacuum. It is supposed that the matter field, initially in its vacuum state, is perturbed by the establishment of an externally generated electromagnetic field, described by the potential $A_\mu(x)$. It will be convenient to assume that the potential vanishes prior to the creation of the field, as well as after the eventual removal of the field, which restricts the otherwise unlimited group of gauge transformations associated with an external electromagnetic field. Indeed, according to this specialization, the function $\Lambda(x)$ that generates a gauge transformation must be constant before the establishment of the field. In placing this constant equal to zero, no further assumption is introduced. We may then characterize the initial matter vacuum state by a unique state vector Ψ_0, without the inconsequential ambiguity associated with gauge transformations. The alteration of the state vector, produced by the external field, is described by

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}=-\frac{1}{c}j_\mu(x)A_\mu(x)\Psi[\sigma]. \tag{2.1}$$

Following the program of I, Section 4, we replace (2.1) by the functional integral equation

$$\Psi[\sigma]=\Psi_0+\frac{i}{\hbar c^2}\int_{-\infty}^{\sigma}j_\mu(x')A_\mu(x')\Psi[\sigma']d\omega', \tag{2.2}$$

which includes the initial condition

$$\Psi[\sigma]\rightarrow\Psi_0,\quad \sigma\rightarrow-\infty, \tag{2.3}$$

and can be solved by successive substitution. We shall be content with the first approximation, which regards the disturbance of the vacuum as small.

$$\Psi[\sigma]=\left(1+\frac{i}{\hbar c^2}\int_{-\infty}^{\sigma}j_\mu(x')A_\mu(x')d\omega'\right)\Psi_0$$
$$=U[\sigma,-\infty]\Psi_0. \tag{2.4}$$

The operator $U[\sigma,-\infty]$ is unitary, to the order of approximation considered. The expectation value of $j_\mu(x)$, computed for the state of the system as modified by the external electromagnetic field, is

$$\langle j_\mu(x)\rangle=(\Psi[\sigma],\,j_\mu(x)\Psi[\sigma])$$
$$=(\Psi_0,\,U^{-1}[\sigma,-\infty]j_\mu(x)U[\sigma,-\infty]\Psi_0)$$
$$=\langle U^{-1}[\sigma,-\infty]j_\mu(x)U[\sigma,-\infty]\rangle_0. \tag{2.5}$$

To the required order of approximation,

$$U^{-1}[\sigma,-\infty]j_\mu(x)U[\sigma,-\infty]$$
$$=j_\mu(x)+\frac{i}{\hbar c^2}\int_{-\infty}^{\sigma}[j_\mu(x),\,j_\nu(x')]A_\nu(x')d\omega', \tag{2.6}$$

whence

$$\langle j_\mu(x)\rangle=\frac{i}{\hbar c^2}\int_{-\infty}^{\sigma}\langle[j_\mu(x),\,j_\nu(x')]\rangle_0A_\nu(x')d\omega'. \tag{2.7}$$

An important test to which this expression should be subjected is that of gauge invariance.

We must require that, in the absence of a real electromagnetic field, no current be induced in the vacuum; that is, $A_\mu(x) = -\partial\Lambda(x)/\partial x_\mu$ must imply $\langle j_\mu(x)\rangle = 0$. On introducing this form for $A_\mu(x)$, we find that

$$\begin{aligned}\langle j_\mu(x)\rangle &= -\frac{i}{\hbar c^2}\int_{-\infty}^{\sigma} d\omega' \frac{\partial}{\partial x_\nu'} \\ &\qquad \times \{\langle [j_\mu(x), j_\nu(x')]\rangle_0 \Lambda(x')\} \\ &= -\frac{i}{\hbar c^2}\int_\sigma d\sigma_\nu' \langle [j_\mu(x), j_\nu(x')]\rangle_0 \Lambda(x') \\ &= -\frac{i}{\hbar c^2}\Lambda(x)\int_\sigma \langle [j_\mu(x), j_\nu(x')]\rangle_0 d\sigma_\nu' \\ &= 0,\end{aligned} \tag{2.8}$$

as required. In the course of this proof we have employed the commutability of all components of the current at two distinct points of a space-like surface, and of a time-like component of the current with j_μ at the same point, as contained in (I, 2.34). The previously discussed requirement that Λ vanish in the remote past is also involved.

In order to construct the vacuum expectation value of the commutator contained in (2.7), we write (see (I, 2.33)):

$$\begin{aligned}[j_\mu(x), j_\nu(x')] &= \frac{ie^2c^2}{2}\{[\psi_\alpha(x), \bar\psi_\beta(x')](\gamma_\nu S(x'-x)\gamma_\mu)_{\beta\alpha} \\ &\qquad - [\psi_\alpha(x'), \bar\psi_\beta(x)](\gamma_\mu S(x-x')\gamma_\nu)_{\beta\alpha}\},\end{aligned} \tag{2.9}$$

whence

$$\begin{aligned}\langle [j_\mu(x), j_\nu(x')]\rangle_0 &= \frac{ie^2c^2}{2} Tr[S^{(1)}(x'-x)\gamma_\mu S(x-x')\gamma_\nu \\ &\qquad - S^{(1)}(x-x')\gamma_\nu S(x'-x)\gamma_\mu].\end{aligned} \tag{2.10}$$

To evaluate this trace, we first remark that the product of any three γ's (more generally, an odd number) has a vanishing trace:

$$Tr\gamma_\mu\gamma_\nu\gamma_\lambda = 0. \tag{2.11}$$

Indeed,

$$\begin{aligned}Tr\gamma_\mu\gamma_\nu\gamma_\lambda &= Tr\tfrac{1}{2}\gamma_\mu\gamma_\nu\gamma_\lambda(\gamma_5\gamma_5+\gamma_5\gamma_5) \\ &= Tr\tfrac{1}{2}(\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_5+\gamma_5\gamma_\mu\gamma_\nu\gamma_\lambda)\gamma_5 = 0,\end{aligned}$$

since γ_5 anti-commutes with all components of γ_μ. Therefore,

$$\begin{aligned}&Tr[S^{(1)}(x'-x)\gamma_\mu S(x-x')\gamma_\nu \\ &\qquad - S^{(1)}(x-x')\gamma_\nu S(x'-x)\gamma_\mu] \\ &= -\frac{\partial\Delta^{(1)}(x-x')}{\partial x_\lambda}\frac{\partial\Delta(x-x')}{\partial x_\sigma} \\ &\qquad \times Tr(\gamma_\lambda\gamma_\mu\gamma_\sigma\gamma_\nu+\gamma_\lambda\gamma_\nu\gamma_\sigma\gamma_\mu) \\ &\quad + \kappa_0^2\Delta^{(1)}(x-x')\Delta(x-x')Tr(\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu)\end{aligned} \tag{2.12}$$

in which the even and odd natures of $\Delta^{(1)}$ and Δ have been employed. Now

$$\begin{aligned}\gamma_\lambda\gamma_\mu\gamma_\sigma\gamma_\nu+\gamma_\lambda\gamma_\nu\gamma_\sigma\gamma_\mu &= 2\delta_{\mu\sigma}\gamma_\lambda\gamma_\nu \\ &\qquad + 2\delta_{\nu\sigma}\gamma_\lambda\gamma_\mu - 2\delta_{\mu\nu}\gamma_\lambda\gamma_\sigma,\end{aligned} \tag{2.13}$$

so that

$$\begin{aligned}&Tr(\gamma_\lambda\gamma_\mu\gamma_\sigma\gamma_\nu+\gamma_\lambda\gamma_\nu\gamma_\sigma\gamma_\mu) \\ &\qquad = 8(\delta_{\mu\sigma}\delta_{\lambda\nu}+\delta_{\nu\sigma}\delta_{\lambda\mu}-\delta_{\mu\nu}\delta_{\lambda\sigma}),\end{aligned} \tag{2.14}$$

and, finally

$$\begin{aligned}&\langle [j_\mu(x), j_\nu(x')]\rangle_0 \\ &= -4ie^2c^2\left[\frac{\partial\Delta(x-x')}{\partial x_\mu}\frac{\partial\Delta^{(1)}(x-x')}{\partial x_\nu}\right. \\ &\qquad + \frac{\partial\Delta(x-x')}{\partial x_\nu}\frac{\partial\Delta^{(1)}(x-x')}{\partial x_\mu} \\ &\qquad - \delta_{\mu\nu}\left(\frac{\partial\Delta(x-x')}{\partial x_\lambda}\frac{\partial\Delta^{(1)}(x-x')}{\partial x_\lambda}\right. \\ &\qquad\qquad \left.\left. + \kappa_0^2\Delta(x-x')\Delta^{(1)}(x-x')\right)\right].\end{aligned} \tag{2.15}$$

In order to simplify further discussion, we shall suppose that the electromagnetic field under consideration does not produce actual electron-positron pairs in the vacuum; that is, we treat only the phenomenon of virtual pair creation. The restriction thereby imposed can be obtained from (2.4). The final state of the matter field, resulting from the establishment and subsequent removal of an electromagnetic field in the vacuum, is given by

$$\Psi[\infty] = \left(1 + \frac{i}{\hbar c^2}\int_{-\infty}^{\infty} j_\mu(x')A_\mu(x')d\omega'\right)\Psi_0, \tag{2.16}$$

which must be simply Ψ_0 if no real pair creation has occurred. Hence,

$$\left[\int_{-\infty}^{\infty} j_\mu(x)A_\mu(x)d\omega\right]\Psi_0=0 \qquad (2.17)$$

is the condition describing the absence of real pair creation events. According to the discussion in I, Section 4, (2.17) will indeed result if the energy and momentum conservation laws cannot be simultaneously obeyed in the course of the pair producing interaction between the electromagnetic field and the fluctuating current in the vacuum. To exploit this limitation, we may rewrite (2.7) as

$$\langle j_\mu(x)\rangle=\frac{i}{\hbar c^2}\int_{-\infty}^{\infty}\langle[j_\mu(x),\, j_\nu(x')]\rangle_0 \times\tfrac{1}{2}(1+\epsilon(x-x'))A_\nu(x')d\omega', \qquad (2.18)$$

where $\epsilon(x)$ is $+1$ or -1 according as x_0 is positive or negative, which is effectively invariant since only time-like intervals $x_\mu - x_\mu'$ occur in (2.18). The condition (2.17) now enables us to replace (2.18) with

$$\langle j_\mu(x)\rangle=\frac{i}{2\hbar c^2}\int_{-\infty}^{\infty}\langle[j_\mu(x),\, j_\nu(x')]\rangle_0 \times\epsilon(x-x')A_\nu(x')d\omega'. \qquad (2.19)$$

The advantage of this form is the possibility of writing

$$\begin{aligned}\langle[j_\mu(x),\, j_\nu(x')]\rangle_0\epsilon(x-x') =8ie^2c^2\bigg[&\frac{\partial\bar{\Delta}(x-x')}{\partial x_\mu}\frac{\partial\Delta^{(1)}(x-x')}{\partial x_\nu} +\frac{\partial\bar{\Delta}(x-x')}{\partial x_\nu}\frac{\partial\Delta^{(1)}(x-x')}{\partial x_\mu} \\ &-\delta_{\mu\nu}\bigg(\frac{\partial\bar{\Delta}(x-x')}{\partial x_\lambda}\frac{\partial\Delta^{(1)}(x-x')}{\partial x_\lambda} +\kappa_0^2\bar{\Delta}(x-x')\Delta^{(1)}(x-x')\bigg)\bigg], \qquad (2.20)\end{aligned}$$

where

$$\bar{\Delta}(x)=-\tfrac{1}{2}\Delta(x)\epsilon(x) \qquad (2.21)$$

shares with $\Delta^{(1)}(x)$ the property of being a function only of $\lambda=-x_\mu^2$. Involved in the relation (2.20) is the fact that

$$\Delta(x)(\partial\epsilon(x)/\partial x_\mu)=0, \qquad (2.22)$$

which is demonstrated by remarking that $\epsilon(x)$ varies only by crossing a space-like surface through the origin, on which $\Delta(x)$ vanishes.

It will now be shown that

$$\langle[j_\mu(x),\, j_\nu(x')]\rangle_0\epsilon(x-x') =8ie^2c^2\left[\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}G(\lambda)-\delta_{\mu\nu}\Box^2G(\lambda)\right], \qquad (2.23)$$

where

$$2\frac{\partial\bar{\Delta}(\lambda)}{\partial\lambda}\frac{\partial\Delta^{(1)}(\lambda)}{\partial\lambda}=\frac{\partial^2G(\lambda)}{\partial\lambda^2}, \qquad (2.24)$$

and $\lambda=-(x_\mu-x_\mu')^2$. The function $G(\lambda)$ is made precise by requiring that it vanish at infinity. Let it first be noted that

$$\begin{aligned}\frac{\partial\bar{\Delta}(x)}{\partial x_\mu}\frac{\partial\Delta^{(1)}(x)}{\partial x_\nu}+\frac{\partial\bar{\Delta}(x)}{\partial x_\nu}\frac{\partial\Delta^{(1)}(x)}{\partial x_\mu} &=4x_\mu x_\nu\frac{\partial^2G(\lambda)}{\partial\lambda^2} \\ &=\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}G(\lambda)+2\delta_{\mu\nu}\frac{\partial G(\lambda)}{\partial\lambda}, \qquad (2.25)\end{aligned}$$

so that

$$\langle[j_\mu(x),\, j_\nu(x')]\rangle_0\epsilon(x-x') =8ie^2c^2\left[\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}G(\lambda)-\delta_{\mu\nu}H(\lambda)\right], \qquad (2.26)$$

where

$$\begin{aligned}H(\lambda)&=\tfrac{1}{2}\Box^2G(\lambda)+2\frac{\partial G(\lambda)}{\partial\lambda}+\kappa_0^2\bar{\Delta}(\lambda)\Delta^{(1)}(\lambda) \\ &=-2\lambda\frac{\partial^2G}{\partial\lambda^2}-2\frac{\partial G(\lambda)}{\partial\lambda}+\kappa_0^2\bar{\Delta}(\lambda)\Delta^{(1)}(\lambda).\end{aligned} \qquad (2.27)$$

The stated simplification of $H(\lambda)$, namely

$$H(\lambda)=\Box^2G(\lambda), \qquad (2.28)$$

can be proven with the aid of the theorem

$$\frac{\partial}{\partial x_\mu}(\langle[j_\mu(x),\, j_\nu(x')]\rangle_0\epsilon(x-x'))=0 \qquad (2.29)$$

since the indicated differentiation, applied to (2.26), yields

$$0=\frac{\partial}{\partial x_\nu}[\Box^2G(\lambda)-H(\lambda)], \qquad (2.30)$$

from which (2.28) follows. To verify (2.29), observe that the left side reduces to

$$\langle[j_\mu(x),\,j_\nu(x')]\rangle_0\frac{\partial\epsilon(x-x')}{\partial x_\mu}.$$

Now $\partial\epsilon(x-x')/\partial x_\mu$ is a non-vanishing time-like vector only if x and x' lie on a space-like surface, and, since a time-like component of the current commutes with j_ν at all points on a space-like surface, the validity of (2.29) becomes evident.

The introduction of the relation (2.23) into the formula (2.19) for the induced current gives, after an integration by parts that employs the vanishing of the external potential in the remote past and future,

$$\begin{aligned}\langle j_\mu(x)\rangle &= 4\frac{e^2}{\hbar}\int_{-\infty}^{\infty}G(\lambda)\\ &\quad\times\left(\Box'^2A_\mu(x')-\frac{\partial}{\partial x_\mu'}\frac{\partial A_\nu(x')}{\partial x_\nu'}\right)d\omega'\\ &= -4\frac{e^2}{\hbar}\int_{-\infty}^{\infty}G(\lambda)\frac{\partial}{\partial x_\nu'}F_{\mu\nu}(x')d\omega' \qquad (2.31)\\ &= -4\frac{e^2}{\hbar c}\int_{-\infty}^{\infty}G(\lambda)J_\mu(x')d\omega',\end{aligned}$$

where $J_\mu(x)$ is the external current generating the electromagnetic field. In this form, the gauge invariance of the theory is made explicit. The induced current depends, not upon the electromagnetic potentials, but rather the field strengths. Our result goes further, however, and states that the induced current at a given space-time point involves only the external current in the vicinity of that point. This has the important consequence that a light wave, propagating at remote distances from its source, induces no current in the vacuum and therefore is undisturbed in its passage through space. There is no light quantum self energy phenomenon akin to that for electrons, as we shall further discuss.

Our last task is the explicit construction of the function $G(\lambda)$. On inserting the integral representations for $\bar{\Delta}(\lambda)$ and $\Delta^{(1)}(\lambda)$ (Eqs. (A.15) and (A.33)):

$$\begin{aligned}\bar{\Delta}(\lambda) &= \frac{1}{8\pi^2}\int_{-\infty}^{\infty}\exp\left(i\lambda\alpha+i\frac{\kappa_0^2}{4\alpha}\right)d\alpha,\\ \Delta^{(1)}(\lambda) &= \frac{i}{4\pi^2}\int_{-\infty}^{\infty}\exp\left(i\lambda\beta+i\frac{\kappa_0^2}{4\beta}\right)\frac{\beta}{|\beta|}d\beta\end{aligned} \qquad (2.32)$$

into (2.24), we obtain

$$\frac{\partial^2G(\lambda)}{\partial\lambda^2} = -\frac{i}{(2\pi)^4}\int_{-\infty}^{\infty}\times\exp\left[i\lambda(\alpha+\beta)+\frac{i\kappa_0^2}{4}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\right]\alpha\beta\frac{\beta}{|\beta|}d\alpha d\beta,$$

whence

$$G(\lambda) = \frac{i}{(2\pi)^4}\int_{-\infty}^{\infty}\exp\left[i\lambda(\alpha+\beta)+\frac{i\kappa_0^2}{4}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\right]\times\frac{\alpha\beta}{(\alpha+\beta)^2}\frac{1}{2}\left(\frac{\alpha}{|\alpha|}+\frac{\beta}{|\beta|}\right)d\alpha d\beta \qquad (2.33)$$

in which a symmetrization with respect to α and β has been performed. It will now be useful to introduce new variables, v and w, defined by

$$\alpha=\frac{\kappa_0^2}{2w}\frac{1}{1-v},\quad \beta=\frac{\kappa_0^2}{2w}\frac{1}{1+v}, \qquad (2.34)$$

which are such that

$$\begin{aligned}G(\lambda) &= \frac{2i}{(4\pi)^4}\kappa_0^4\int_{-\infty}^{\infty}\exp\left(iw+i\frac{\kappa_0^2\lambda}{w(1-v^2)}\right)\\ &\quad\times\frac{1}{2}\left(\frac{1+v}{|1+v|}+\frac{1-v}{|1-v|}\right)\frac{dv}{1-v^2}\frac{dw}{w^3}\\ &= \frac{2i}{(4\pi)^4}\kappa_0^4\int_{-1}^{1}\frac{dv}{1-v^2}\int_{-\infty}^{\infty}\frac{dw}{w^3}\\ &\quad\times\exp\left(iw+i\frac{\kappa_0^2\lambda}{w(1-v^2)}\right).\end{aligned} \qquad (2.35)$$

The integral representation

$$\int(dk)\exp(ik_\mu(x_\mu-x_\mu'))\exp\left(i\frac{k_\mu^2}{4\kappa_0^2}w(1-v^2)\right) = i(4\pi)^2\kappa_0^4\frac{\exp\left(i\frac{\kappa_0^2\lambda}{w(1-v^2)}\right)}{w|w|(1-v^2)^2} \qquad (2.36)$$

then transforms (2.35) into

$$G(\lambda)=\frac{8}{(4\pi)^6}\int(dk)\exp(ik_\mu(x_\mu-x_\mu'))\times\int_0^1(1-v^2)dv\int_0^\infty\frac{dw}{w}\times\cos\left(1+\frac{k_\mu^2}{4\kappa_0^2}(1-v^2)\right)w,\quad(2.37)$$

which can be followed by an integration by parts with respect to v, according to

$$\int_0 d\left(v-\frac{v^3}{3}\right)\int_0^\infty\frac{dw}{w}\cos\left(1+\frac{k_\mu^2}{4\kappa_0^2}(1-v^2)\right)w$$
$$=\frac{2}{3}\int_0^\infty\frac{\cos w}{w}dw-\frac{k_\mu^2}{2\kappa_0^2}\int_0^1\left(1-\frac{v^2}{3}\right)v^2dv\times\int_0^\infty dw\,\sin\left(1+\frac{k_\mu^2}{4\kappa_0^2}(1-v^2)\right)w.\quad(2.38)$$

The first w integral is logarithmically divergent at the origin. On introducing a lower limit, w_0, we obtain

$$\frac{2}{3}\log\frac{1}{\gamma w_0}-\frac{k_\mu^2}{2\kappa_0^2}P\int_0^1\frac{1-(v^2/3)}{1+(k_\mu^2/4\kappa_0^2)(1-v^2)}v^2dv$$

as the value of the integral (2.38), where $\gamma=1.781$. The insertion of this result into (2.37) yields

$$G(\lambda)=\frac{1}{48\pi^2}\log\frac{1}{\gamma w_0}\delta(x-x')+\frac{4}{(4\pi)^6}\frac{1}{\kappa_0^2}\Box^2\int_0^1\left(1-\frac{v^2}{3}\right)v^2dvP\times\int(dk)\frac{\exp(ik_\mu(x_\mu-x_\mu'))}{1+(k_\mu^2/4\kappa_0^2)(1-v^2)}\quad(2.39)$$

in which it has been noticed that the operator $\Box^2$ is equivalent to multiplication by $-k_\mu^2$ in its effect on $\exp(ik_\mu x_\mu)$, and that

$$\frac{1}{(2\pi)^4}\int\exp(ik_\mu x_\mu)(dk)=\delta(x)=\delta(x_0)\delta(x_1)\delta(x_2)\delta(x_3).\quad(2.40)$$

The Fourier integral contained in the second term of (2.39) can be recognized as related to that of the function $\bar{\Delta}(x)$ (Eq. (A.10)):

$$\bar{\Delta}(x)=\frac{1}{(2\pi)^4}P\int\frac{\exp(ik_\mu x_\mu)}{k_\mu^2+\kappa_0^2}(dk).\quad(2.41)$$

Indeed,

$$\frac{1}{(2\pi)^4}P\int\frac{\exp(ik_\mu x_\mu)}{1+(k_\mu^2/4\kappa_0^2)(1-v^2)}(dk)=\frac{16\kappa_0^2}{(1-v^2)^2}\bar{\Delta}\left(\frac{2}{(1-v^2)^{\frac{1}{2}}}x\right),\quad(2.42)$$

so that

$$G(\lambda)=\frac{1}{48\pi^2}\log\frac{1}{\gamma w_0}\delta(x-x')+\frac{1}{4\pi^2}\Box^2\int_0^1\bar{\Delta}\left(\frac{2}{(1-v^2)^{\frac{1}{2}}}(x-x')\right)\times\frac{1-\frac{1}{3}v^2}{(1-v^2)^2}v^2dv.\quad(2.43)$$

The expression for the induced current that is obtained from this form for $G(\lambda)$ is

$$\langle j_\mu(x)\rangle=-\frac{\alpha}{3\pi}\log\frac{1}{\gamma\omega_0}J_\mu(x)-\frac{4}{\pi}\alpha\int d\omega'\Box^2\times\int_0^1\bar{\Delta}\left(\frac{2}{(1-v^2)^{\frac{1}{2}}}(x-x')\right)\times\frac{1-\frac{1}{3}v^2}{(1-v^2)^2}v^2dvJ_\mu(x'),\quad(2.44)$$

where $\alpha=e^2/4\pi\hbar c$ is the fine structure constant. The current induced at a given point is thus exhibited in two parts: a logarithmically divergent multiple of the external current at that point, and a finite contribution involving the external current in the vicinity of the given point. The first part reduces the strength of the external current by a constant factor and hence produces an unobservable charge renormalization, as discussed in I. The second part of (2.44) is therefore the physically significant induced current.

An alternative form for the latter is[2]

$$\langle j_\mu(x)\rangle = -\frac{4}{\pi}\alpha\int d\omega'\int_0^1 \bar{\Delta}\left(\frac{2}{(1-v^2)^{\frac{1}{2}}}(x-x')\right) \times\frac{1-\frac{1}{3}v^2}{(1-v^2)^2}v^2dv\Box'^2J_\mu(x'). \quad (2.45)$$

If the external current varies sufficiently slowly, the relevant unit of length being $1/\kappa_0=\hbar/m_0c$, one can obtain a series in ascending powers of $\Box^2$ applied to $J_\mu(x)$. This may be done by continued application of the relation:

$$\bar{\Delta}\left(\frac{2}{(1-v^2)^{\frac{1}{2}}}x\right)=\frac{(1-v^2)^2}{16\kappa_0^2}\delta(x) +\frac{1-v^2}{4\kappa_0^2}\Box^2\bar{\Delta}\left(\frac{2}{(1-v^2)^{\frac{1}{2}}}x\right). \quad (2.46)$$

Thus

$$\begin{aligned}\langle j_\mu(x)\rangle &= -\frac{\alpha}{15\pi}\frac{1}{\kappa_0^2}\Box^2J_\mu(x)-\frac{\alpha}{\pi}\frac{1}{\kappa_0^2}\int d\omega' \times\int_0^1\bar{\Delta}\left(\frac{2}{(1-v^2)^{\frac{1}{2}}}(x-x')\right) \times\frac{1-\frac{1}{3}v^2}{1-v^2}v^2dv\Box'^2\Box'^2J_\mu(x') \\ &= -\frac{\alpha}{15\pi}\frac{1}{\kappa_0^2}\Box^2J_\mu(x) -\frac{\alpha}{140\pi}\left(\frac{1}{\kappa_0^2}\Box^2\right)^2J_\mu(x)-\cdots.\end{aligned} \quad (2.47)$$

If, however, the first term[3] is not an adequate approximation, the series will usually be inconvenient and recourse must be had to the original integral expression.

A particular case of importance is that of a time independent external charge or current distribution. In this situation, (2.45) can be specialized to

$$\langle j_\mu(x)\rangle = -\frac{4}{\pi}\alpha\int d\tau'\int_{-\infty}^{\infty}dx_0'\int_0^1 \times\bar{\Delta}\left(\frac{2}{(1-v^2)^{\frac{1}{2}}}(\mathbf{r}-\mathbf{r}'),\ \frac{2}{(1-v^2)^{\frac{1}{2}}}(x_0-x_0')\right) \times\frac{1-\frac{1}{3}v^2}{(1-v^2)^2}v^2dv\nabla'^2J_\mu(\mathbf{r}'), \quad (2.48)$$

where $d\tau'$ here denotes a three-dimensional volume element. Now

$$G(\mathbf{r})=\int_{-\infty}^{\infty}\bar{\Delta}(\mathbf{r},x_0)dx_0 \quad (2.49)$$

obeys the differential equation

$$(\nabla^2-\kappa_0^2)G(\mathbf{r})=-\delta(\mathbf{r}) \quad (2.50)$$

and is therefore the three-dimensional Green's function:

$$G(\mathbf{r})=\frac{e^{-\kappa_0 r}}{4\pi r}. \quad (2.51)$$

Accordingly,[4]

$$\begin{aligned}\langle j_\mu(x)\rangle &= -\frac{\alpha}{4\pi^2}\int d\tau'\int_0^1\frac{\exp\left(-\frac{2\kappa_0}{(1-v^2)^{\frac{1}{2}}}|\mathbf{r}-\mathbf{r}'|\right)}{|\mathbf{r}-\mathbf{r}'|} \times\frac{1-\frac{1}{3}v^2}{1-v^2}v^2dv\nabla'^2J_\mu(\mathbf{r}') \\ &= -\frac{\alpha}{6\pi^2}\int K(\mathbf{r}-\mathbf{r}')\nabla'^2J_\mu(\mathbf{r}')d\tau',\end{aligned} \quad (2.52)$$

with

$$K(\mathbf{r})=\int_1^\infty\frac{e^{-2\kappa_0 r\xi}}{r}\left(1+\frac{1}{2\xi^2}\right)\frac{(\xi^2-1)^{\frac{1}{2}}}{\xi^2}d\xi. \quad (2.53)$$

We shall be content to record the asymptotic forms of $K(r)$, which can be expressed in terms of the Hankel function of imaginary argument, $K_0(2\kappa_0 r)$, and associated functions:[5]

$$\begin{aligned}K(\mathbf{r}) &= \frac{1}{r}\left(\log\frac{1}{\gamma\kappa_0 r}-\frac{5}{6}\right), \quad \kappa_0 r\ll 1 \\ K(\mathbf{r}) &= \frac{3\pi^{\frac{1}{2}}}{8r}\frac{e^{-2\kappa_0 r}}{(\kappa_0 r)^{\frac{3}{2}}}, \quad \kappa_0 r\gg 1.\end{aligned} \quad (2.54)$$

[2] Formulae equivalent to (2.45) have been given by R. Serber, Phys. Rev. **48**, 49 (1935).

[3] The original discussion of the polarization of the vacuum were confined to this term. P. A. M. Dirac, 7e Conseil Solvay, 203 (1934); W. Heisenberg, Zeits. f. Physik **90**, 209 (1934).

[4] An equivalent result has been derived by E. A. Uehling, Phys. Rev. **48**, 55 (1935).

[5] W. Pauli and M. E. Rose, Phys. Rev. **49**, 462 (1936).

3. THE SELF-ENERGY OF THE ELECTRON

The second problem to be treated is the modification of the matter field properties arising from its interaction with the vacuum fluctuations of the electromagnetic field. The coupling between the fields can be described in two equivalent ways; either employing the complete electromagnetic four-vector potential, together with a supplementary condition

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}=-\frac{1}{c}j_\mu(x)A_\mu(x)\Psi[\sigma], \quad (3.1a)$$

$$\left(\frac{\partial A_\mu(x)}{\partial x_\mu}-\int_\sigma D(x-x')\frac{1}{c}j_\mu(x')d\sigma_\mu'\right)\Psi[\sigma]=0, \quad (3.1b)$$

or using the transverse four-vector potential, which requires no explicit use of the supplementary condition

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}=\left[-\frac{1}{c}j_\mu(x)\,\mathfrak{A}_\mu(x)-\frac{1}{c^2}\int_\sigma\left(\frac{1}{2}\frac{\partial\mathfrak{D}(x-x')}{\partial x_\mu}+n_\mu n_\nu\frac{\partial\mathfrak{D}(x-x')}{\partial x_\nu}\right)\times j_\mu(x)j_\lambda(x')d\sigma_\lambda'\right]\Psi[\sigma]. \quad (3.2)$$

In discussing either of these equations of motion for $\Psi[\sigma]$, we shall employ a perturbation method based on the weakness of the coupling between the two fields, as measured by the smallness of $\alpha=e^2/4\pi\hbar c=1/137$. Physical quantities will accordingly be classified as to order of magnitude, depending upon the power of e, or better $\alpha^{\frac{1}{2}}$, which they involve.

To zero order, there is no interaction between the fields, and the state vector $\Psi[\sigma]$ is constant. The first-order coupling between the two fields corresponds to the emission or absorption of a light quantum by a free electron, or in the course of creation or annihilation of a pair. It is important that all such processes are virtual; that is, in consequence of the impossibility of simultaneously satisfying energy and momentum conservation laws, a free electron cannot emit or absorb a light quantum, nor can a light quantum create a pair, or a pair annihilate with the emission of a single quantum. Hence first-order interaction effects have no direct physical significance, but only exhibit themselves to the second order in such processes as the virtual emission and subsequent absorption of a light quantum by the matter field, thus producing the interaction between different particles and the self-energy of a single particle. We shall therefore attempt to construct an equation of motion for $\Psi[\sigma]$ from which the first order interaction term has been eliminated and replaced by the second order couplings which it generates.

In order to carry out this program, we must exhibit the first-order solution of the equation of motion for $\Psi[\sigma]$. As in the discussion of vacuum polarization, the required solution, of (3.2) say, is given by

$$\Psi[\sigma]\cong\left(1+\frac{i}{\hbar c^2}\int_{-\infty}^{\sigma}j_\mu(x')\,\mathfrak{A}_\mu(x')d\omega'\right)\Psi, \quad (3.3)$$

where Ψ is the state vector in the absence of interaction. Although this solution has been chosen to fit a boundary condition as $\sigma\rightarrow-\infty$, the absence of any real first-order effect, as expressed by

$$\int_{-\infty}^{\infty}j_\mu(x)\,\mathfrak{A}_\mu(x)d\omega=0, \quad (3.4)$$

enables (3.3) to be rewritten:

$$\Psi[\sigma]\cong(1-iS[\sigma])\Psi$$
$$S[\sigma]=-\frac{1}{2\hbar c^2}\int_{-\infty}^{\infty}j_\mu(x')\,\mathfrak{A}_\mu(x')\epsilon[\sigma,\sigma']d\omega', \quad (3.5)$$

in which form there is no distinction between past and future.

The operator $1-iS[\sigma]$ is unitary only to first order. In order that the unitary property be valid to the second order, the operator can be extended to

$$1-iS[\sigma]-\tfrac{1}{2}(S[\sigma])^2, \quad (3.6)$$

which in turn may be replaced by any rigorous unitary operator that agrees with (3.6) to the desired degree of approximation. The simplest choice of such an operator is $e^{-iS[\sigma]}$. Accordingly, we introduce the state vector transformation

$$\Psi[\sigma]\rightarrow e^{-iS[\sigma]}\Psi[\sigma] \quad (3.7)$$

in which the new state vector varies only in response to second-order interactions. The new equation of motion, replacing (3.2), is

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}+i\hbar c e^{iS[\sigma]}\frac{\delta e^{-iS[\sigma]}}{\delta\sigma(x)}\Psi[\sigma]$$
$$=\bigg[-e^{iS[\sigma]}\frac{1}{c}j_\mu(x)\,\mathfrak{A}_\mu(x)e^{-iS[\sigma]}$$
$$-\frac{1}{c^2}\int_\sigma\left(\frac{1}{2}\frac{\partial\mathfrak{D}(x-x')}{\partial x_\mu}+n_\mu n_\nu\frac{\partial\mathfrak{D}(x-x')}{\partial x_\nu}\right)$$
$$\times j_\mu(x)j_\lambda(x')d\sigma_\lambda'\bigg]\Psi[\sigma], \quad (3.8)$$

in which corrections to the generalized Coulomb term have been discarded, since we shall consistently retain only second-order terms. We are required to evaluate

$$e^{iS[\sigma]}\frac{\delta e^{-iS[\sigma]}}{\delta\sigma(x)}=-i\frac{\delta S[\sigma]}{\delta\sigma(x)}$$
$$+\frac{1}{2}\left[S[\sigma],\frac{\delta S[\sigma]}{\delta\sigma(x)}\right]+\cdots \quad (3.9)$$

and

$$e^{iS[\sigma]}j_\mu(x)\,\mathfrak{A}_\mu(x)e^{-iS[\sigma]}=j_\mu(x)\,\mathfrak{A}_\mu(x)$$
$$+i[S[\sigma],j_\mu(x)\,\mathfrak{A}_\mu(x)]+\cdots. \quad (3.10)$$

Now $S[\sigma]$, as defined in (3.5), satisfies the equation of motion

$$\hbar c\frac{\delta S[\sigma]}{\delta\sigma(x)}=-\frac{1}{c}j_\mu(x)\,\mathfrak{A}_\mu(x), \quad (3.11)$$

whence

$$i\hbar c e^{iS[\sigma]}\frac{\delta e^{-iS[\sigma]}}{\delta\sigma(x)}+e^{iS[\sigma]}\frac{1}{c}j_\mu(x)\,\mathfrak{A}_\mu(x)e^{-iS[\sigma]}$$
$$=\frac{i}{2}\left[S[\sigma]\ \frac{1}{c}j_\mu(x)\,\mathfrak{A}_\mu(x)\right]$$
$$=\frac{i}{4\hbar c^3}\int_{-\infty}^{\infty}[j_\mu(x)\,\mathfrak{A}_\mu(x),\,j_\nu(x')\,\mathfrak{A}_\nu(x')]$$
$$\times\epsilon[\sigma,\sigma']d\omega', \quad (3.12)$$

and

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}=\bigg[-\frac{i}{4\hbar c^3}\int[j_\mu(x)\,\mathfrak{A}_\mu(x),$$
$$j_\nu(x')\,\mathfrak{A}_\nu(x')]\epsilon[\sigma,\sigma']d\omega'$$
$$-\frac{1}{c^2}\int_\sigma\left(\frac{1}{2}\frac{\partial\mathfrak{D}(x-x')}{\partial x_\mu}+n_\mu n_\nu\frac{\partial\mathfrak{D}(x-x')}{\partial x_\nu}\right)$$
$$\times j_\mu(x)j_\lambda(x')d\sigma_\lambda'\bigg]\Psi[\sigma]. \quad (3.13)$$

The same operations can be carried out with Eq. (3.1a); in particular, the functional $S[\sigma]$ occurring in the unitary transformation (3.7) has the same form as in (3.5), but with $A_\mu(x)$ replacing $\mathfrak{A}_\mu(x)$. The result of the transformation is, evidently,

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}=\bigg[-\frac{i}{4\hbar c^3}\int[j_\mu(x)A_\mu(x),\,j_\nu(x')A_\nu(x')]$$
$$\times\epsilon(x-x')d\omega'\bigg]\Psi[\sigma]. \quad (3.14)$$

However, we must also consider the supplementary condition (3.1b), which becomes:

$$\bigg[e^{iS[\sigma]}\frac{\partial A_\mu(x)}{\partial x_\mu}e^{-iS[\sigma]}-\frac{1}{c}\int_\sigma D(x-x')e^{iS[\sigma]}$$
$$\times j_\mu(x')e^{-iS[\sigma]}d\sigma_\mu'\bigg]\Psi[\sigma]=0. \quad (3.15)$$

In order to simplify

$$e^{iS[\sigma]}\frac{\partial A_\mu(x)}{\partial x_\mu}e^{-iS[\sigma]}=\frac{\partial A_\mu(x)}{\partial x_\mu}+i\left[S[\sigma],\frac{\partial A_\mu(x)}{\partial x_\mu}\right]$$
$$-\frac{1}{2}\left[S[\sigma],\left[S[\sigma],\frac{\partial A_\mu(x)}{\partial x_\mu}\right]\right]+\cdots,$$

observe that

$$i\left[S[\sigma],\frac{\partial A_\mu(x)}{\partial x_\mu}\right]$$
$$=\frac{i}{2\hbar c^2}\int\left[\frac{\partial A_\mu(x)}{\partial x_\mu},A_\nu(x')\right]j_\nu(x')\epsilon(x-x')d\omega'$$
$$=\frac{1}{2c}\int\frac{\partial}{\partial x_\nu'}(D(x-x')j_\nu(x'))\epsilon(x-x')d\omega'$$
$$=\frac{1}{c}\int_\sigma D(x-x')j_\nu(x')d\sigma_\nu'. \quad (3.16)$$

The first-order terms are thereby removed from the supplementary condition, which now reads

$$\left[\frac{\partial A_\mu(x)}{\partial x_\mu}-\frac{i}{4\hbar c^3}\int_\sigma d\sigma_\mu'\int d\omega'' D(x-x')\right.$$
$$\left.\times[j_\mu(x'),j_\nu(x'')]A_\nu(x'')\epsilon[\sigma,\sigma'']\right]\Psi[\sigma]=0. \quad (3.17)$$

The commutator contained in (3.14) can be simplified in the following manner:

$$\begin{aligned}[j_\mu(x)A_\mu(x),\, j_\nu(x')A_\nu(x')]\\ &=\tfrac{1}{2}[A_\mu(x),\, A_\nu(x')]\{j_\mu(x),\, j_\nu(x')\}\\ &\quad+\tfrac{1}{2}[j_\mu(x),\, j_\nu(x')]\{A_\mu(x),\, A_\nu(x')\}\\ &=\frac{i\hbar c}{2}\delta_{\mu\nu}D(x-x')\{j_\mu(x),\, j_\nu(x')\}\\ &\quad+\tfrac{1}{2}[j_\mu(x),\, j_\nu(x')]\{A_\mu(x),\, A_\nu(x')\},\end{aligned} \quad (3.18)$$

which brings the equation of motion into the form

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}=\left[-\frac{1}{4c^2}\int\{j_\mu(x),\, j_\mu(x')\}\bar{D}(x-x')d\omega'\right.$$
$$-\frac{i}{8\hbar c^3}\int[j_\mu(x),\, j_\nu(x')]\{A_\mu(x),\, A_\nu(x')\}$$
$$\left.\times\epsilon(x-x')d\omega'\right]\Psi[\sigma]. \quad (3.19)$$

Both terms have an elementary interpretation, describing the self-action of each field through the intermediary of the other field. First note that the operator replacing $A_\mu(x)$, in consequence of the state vector transformation, is

$$\begin{aligned}&e^{iS[\sigma]}A_\mu(x)e^{-iS[\sigma]}\\ &=A_\mu(x)+i[S[\sigma],\, A_\mu(x)]+\cdots\\ &=A_\mu(x)+\frac{1}{c}\int\bar{D}(x-x')j_\mu(x')d\omega'+\cdots.\end{aligned} \quad (3.20)$$

The additional term thus produced,

$$\delta A_\mu(x)=\frac{1}{c}\int\bar{D}(x-x')j_\mu(x')d\omega', \quad (3.21)$$

represents the electromagnetic field induced by the current in virtue of the first-order coupling. This potential satisfies

$$\begin{aligned}\Box^2\delta A_\mu(x)&=-\frac{1}{c}j_\mu(x),\\ \frac{\partial}{\partial x_\mu}\delta A_\mu(x)&=0,\end{aligned} \quad (3.22)$$

and is simply half the sum of advanced and retarded potentials classically ascribed to the current distribution $j_\mu(x)$. Similarly,

$$\begin{aligned}e^{iS[\sigma]}j_\mu(x)e^{-iS[\sigma]}&=j_\mu(x)+i[S[\sigma],\, j_\mu(x)]+\cdots\\ &=j_\mu(x)+\delta j_\mu(x)\end{aligned} \quad (3.23)$$

where

$$\delta j_\mu(x)=\frac{i}{2\hbar c^2}\int[j_\mu(x),\, j_\nu(x')]A_\nu(x')$$
$$\times\epsilon(x-x')d\omega' \quad (3.24)$$

is the current induced by the electromagnetic field. It is the vacuum expectation value of this current that was considered in the previous section. We now observe that (3.19) may be written

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}=\left[-\frac{1}{4c}\{j_\mu(x),\, \delta A_\mu(x)\}\right.$$
$$\left.-\frac{1}{4c}\{\delta j_\mu(x),\, A_\mu(x)\}\right]\Psi[\sigma], \quad (3.25)$$

in which the two terms evidently represent the interaction of the current with the electromagnetic field generated by the current, and of the electromagnetic field with the current induced by the field. The factor of ½ (other than the ½ accompanying the symmetrization of the products) is that inevitably associated with the self-action of a system.

The equation of motion (3.13) can be given an analogous interpretation except that the interaction between the current and the field generated by the current occurs in two parts, associated with the transverse and longitudinal potentials of the current distribution. However, this more involved representation of the field differs from (3.21) only by a gauge transformation. The transverse potential induced by the current is

$$\delta \mathfrak{A}_\mu(x) = \frac{i}{2\hbar c^2}\int [\mathfrak{A}_\mu(x),\ \mathfrak{A}_\nu(x')] j_\nu(x')\epsilon[\sigma,\sigma']d\omega'$$
$$= \frac{1}{c}\int \bar{D}(x-x')j_\mu(x')d\omega'$$
$$+\frac{1}{2c}\int\left[\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}+\left(n_\mu\frac{\partial}{\partial x_\nu}+n_\nu\frac{\partial}{\partial x_\mu}\right)n_\lambda\frac{\partial}{\partial x_\lambda}\right]\mathfrak{D}(x-x')$$
$$\times j_\nu(x')\epsilon[\sigma,\sigma']d\omega'. \quad (3.26)$$

Now

$$\frac{1}{2}\int\left[\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}+n_\mu\frac{\partial}{\partial x_\nu}n_\lambda\frac{\partial}{\partial x_\lambda}\right]$$
$$\times\mathfrak{D}(x-x')j_\nu(x')\epsilon[\sigma,\sigma']d\omega'$$
$$=-\frac{1}{2}\int\frac{\partial}{\partial x_\nu'}\left[\left(\frac{\partial}{\partial x_\mu}+n_\mu n_\lambda\frac{\partial}{\partial x_\lambda}\right)\times\mathfrak{D}(x-x')j_\nu(x')\right]\epsilon[\sigma,\sigma']d\omega'$$
$$=-\int_\sigma\left(\frac{\partial}{\partial x_\mu}+n_\mu n_\nu\frac{\partial}{\partial x_\nu}\right)\mathfrak{D}(x-x')j_\lambda(x')d\sigma_\lambda'. \quad (3.27)$$

and

$$\frac{1}{2}\int n_\nu\frac{\partial}{\partial x_\mu}n_\lambda\frac{\partial}{\partial x_\lambda}\mathfrak{D}(x-x')j_\nu(x')\epsilon[\sigma,\sigma']d\omega'$$
$$=\frac{\partial}{\partial x_\mu}\left[\frac{1}{2}\int n_\lambda\frac{\partial}{\partial x_\lambda}\mathfrak{D}(x-x')n_\nu j_\nu(x')\epsilon[\sigma,\sigma']d\omega'\right]$$
$$+\frac{1}{2}\int n_\lambda\frac{\partial}{\partial x_\lambda}\mathfrak{D}(x-x')j_\nu(x')n_\nu\frac{\partial}{\partial x_\mu'}$$
$$\times\epsilon[\sigma,\sigma']d\omega'. \quad (3.28)$$

It is a consequence of the time-like nature of the gradient of $\epsilon[\sigma,\sigma']$ that

$$n_\nu\frac{\partial}{\partial x_\mu'}\epsilon[\sigma,\sigma']=n_\mu\frac{\partial}{\partial x_\nu'}\epsilon[\sigma,\sigma'],$$

whence the second term of the right side of (3.28) becomes

$$-\int_\sigma n_\mu n_\nu\frac{\partial}{\partial x_\nu}\mathfrak{D}(x-x')j_\lambda(x')d\sigma_\lambda'.$$

Hence,

$$\delta\mathfrak{A}_\mu(x)=\delta A_\mu(x)-\frac{1}{c}\int_\sigma\left(\frac{\partial}{\partial x_\mu}+n_\mu n_\nu\frac{\partial}{\partial x_\nu}\right)$$
$$\times\mathfrak{D}(x-x')j_\lambda(x')d\sigma_\lambda'$$
$$+\frac{\partial}{\partial x_\mu}\left[\frac{1}{2}\int n_\lambda\frac{\partial}{\partial x_\lambda}\mathfrak{D}(x-x')\times n_\nu j_\nu(x')\epsilon[\sigma,\sigma']d\omega'\right]. \quad (3.29)$$

The gradient term can be completely eliminated, in the approximation of retaining only second order quantities, by a suitable gauge transformation on the state vector. The second term of $\delta\mathfrak{A}_\mu(x)$ exactly cancels the Coulomb coupling expression, and we are left with

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}=\left[-\frac{1}{4c}\{j_\mu(x),\ \delta A_\mu(x)\}-\frac{1}{4c}\{\delta j_\mu(x),\ \mathfrak{A}_\mu(x)\}\right]\Psi[\sigma] \quad (3.30)$$

as the simplified form of (3.13). Here $\delta j_\mu(x)$ is given by (3.24) but with $\mathfrak{A}_\mu(x)$ replacing $A_\mu(x)$. It will be evident from this discussion that the separate consideration of longitudinal and transverse fields is an inadvisable complication in the treatment of virtual light quantum processes.

The current induced by the electromagnetic field is naturally divided into two parts, that existing in the absence of any charged particles,

$$(\delta j_\mu(x))_0=\frac{i}{2\hbar c^2}\int\langle[j_\mu(x),\ j_\nu(x')]\rangle_0$$
$$\times\mathfrak{A}_\nu(x')\epsilon[\sigma,\sigma']d\omega', \quad (3.31)$$

and that specifically associated with the presence of matter,

$$(\delta j_\mu(x))_1=\frac{i}{2\hbar c^2}\int\{[j_\mu(x),\ j_\nu(x')]$$
$$-\langle[j_\mu(x),\ j_\nu(x')]\rangle_0\}\mathfrak{A}_\nu(x')\epsilon[\sigma,\sigma']d\omega'. \quad (3.32)$$

Were the vacuum induced current different from zero, Eq. (3.30) would contain a term describing modifications in the properties of the electromagnetic field, without the presence of charged particles.[6] Such a light quantum self-energy effect does not in fact exist, since no current is induced in the vacuum by a light wave, as we have shown in the previous section. However, note that while

$$\square^2 \mathfrak{A}_\mu(x) - \frac{\partial}{\partial x_\mu}\left(\frac{\partial \mathfrak{A}_\nu(x)}{\partial x_\nu}\right) = 0,$$

as required in the proof, further discussion is required for $A_\mu(x)$, since the supplementary condition is involved in the treatment of $\partial A_\mu(x)/\partial x_\mu$. Now

$$\begin{aligned}\{(\delta j_\mu(x))_0, A_\mu(x)\}\Psi[\sigma] \\ = \frac{i}{2\hbar c^2}\int \langle[j_\mu(x), j_\nu(x')]\rangle_0[A_\mu(x)A_\nu(x') \\ +A_\nu(x')A_\mu(x)]\epsilon(x-x')d\omega'\Psi[\sigma] \\ = \frac{i}{\hbar c^2}A_\mu(x)\int\langle[j_\mu(x), j_\nu(x')]\rangle_0 \\ \times A_\nu(x')\epsilon(x-x')d\omega'\Psi[\sigma] \\ +\frac{1}{2c}\int\langle[j_\mu(x), j_\mu(x')]\rangle_0\epsilon(x-x') \\ \times D(x-x')d\omega'\Psi[\sigma]. \quad (3.33)\end{aligned}$$

In the first term on the right side of Eq. (3.33), $\partial A_\nu(x')/\partial x_\nu'$ operates directly on $\Psi[\sigma]$, permitting the supplementary condition (3.17) to be invoked and is effectively equal to zero since a fourth order quantity is to be neglected. The second term of (3.33) also vanishes since $\langle[j_\mu(x), j_\mu(x')]\rangle_0\epsilon(x-x')$ is an even function, while $D(x-x')$ is an odd function of $x-x'$. Therefore,

$$\{(\delta j_\mu(x))_0, A_\mu(x)\}\Psi[\sigma] = 0. \quad (3.34)$$

It is convenient to divide the interaction of the electromagnetic field with the current that it induces,

$$-\frac{1}{4c}\{(\delta j_\mu(x))_1, \mathfrak{A}_\mu(x)\} = -\frac{i}{8\hbar c^3}\int[j_\mu(x), j_\nu(x')]_1 \times\{\mathfrak{A}_\mu(x), \mathfrak{A}_\nu(x')\}\epsilon(x-x')d\omega', \quad (3.35)$$

into two parts, of which one is associated with the vacuum fluctuations of the electromagnetic field,

$$-\frac{1}{4c}\{(\delta j_\mu(x))_1, \mathfrak{A}_\mu(x)\}_0 = -\frac{i}{8\hbar c^3}\int[j_\mu(x), j_\nu(x')]_1 \times\langle\{\mathfrak{A}_\mu(x), \mathfrak{A}_\nu(x')\}\rangle_0\epsilon(x-x')d\omega', \quad (3.36)$$

and the other exists only in the presence of actual light quanta:

$$-\frac{1}{4c}\{(\delta j_\mu(x))_1, \mathfrak{A}_\mu(x)\}_1 = -\frac{i}{8\hbar c^3}\int[j_\mu(x), j_\nu(x')]_1 \times\{\mathfrak{A}_\mu(x), \mathfrak{A}_\nu(x')\}_1\epsilon(x-x')d\omega'. \quad (3.37)$$

In these formulae, the subscript one indicates the difference between the quantity and its vacuum expectation value. The second part of the interaction, Eq. (3.37), describes the real coupling between matter and radiation, as exhibited in such processes as the scattering of a light quantum by an electron, and the two quantum annihilation of an electron-position pair. The first part, Eq. (3.36), contains only the dynamical variables of the matter field and constitutes a portion of the electron self-energy.

A similar decomposition can be performed with

$$-\frac{1}{4c}\{(\delta j_\mu(x))_1, A_\mu(x)\} = -\frac{i}{8\hbar c^2}\int[j_\mu(x), j_\nu(x')]_1 \times\{A_\mu(x), A_\nu(x')\}\epsilon(x-x')d\omega' \quad (3.38)$$

except that the definition of the vacuum for the field described by $A_\mu(x)$ requires a slight discussion. Evidently a statement concerning the vacuum state of the longitudinal fields is meaningless since these fields are completely eliminated by the supplementary condition. However, it is certainly permissible to adopt a conventional definition that unifies the treatment of the longitudinal and transverse fields, with the full knowledge that the eventual elimination of the longitudinal fields will deprive the particular convention of any physical content. For this reason, the definition of the vacuum (1.34) may be extended by the conventions

$$\Lambda^{(+)}(x)\Psi_0 = 0, \quad \Lambda'^{(+)}(x)\Psi_0 = 0, \quad (3.39)$$

thus yielding

$$A_\mu^{(+)}(x)\Psi_0 = 0 \quad (3.40)$$

[6] A logarithmically divergent term of this type was obtained by W. Heisenberg (see reference 3).

as the natural definition of the vacuum in a treatment that employs the complete four-vector potential. Expectation values of quadratic forms can be computed as before:

$$\begin{aligned}\langle A_\mu(x)A_\nu(x')+A_\nu(x')A_\mu(x)\rangle_0 &= -i[A_\mu^{(1)}(x),\, A_\nu(x')] \\ &= \hbar c\delta_{\mu\nu}D^{(1)}(x-x').\end{aligned} \tag{3.41}$$

In particular,

$$-\frac{1}{4c}\{(\delta j_\mu(x))_1,\, A_\mu(x)\}_0=-\frac{i}{8c^2}\int[j_\mu(x),j_\mu(x')]_1 \times\epsilon(x-x')D^{(1)}(x-x')d\omega'. \tag{3.42}$$

That nothing of a physical nature has been added by adopting the vacuum definition (3.40) can be made more convincing by proving the equivalence, to within a gauge transformation, of the two expressions (3.36) and (3.42), involving their respective definitions of the vacuum. Now

$$\begin{aligned}-\frac{1}{4c}\{(\delta j_\mu(x))_1,\, \mathcal{A}_\mu(x)\}_0 &= -\frac{1}{4c}\{(\delta j_\mu(x))_1,\, A_\mu(x)\}_0 \\ &+\frac{i}{8c^2}\int[j_\mu(x),\, j_\nu(x')]_1\bigg(\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu} \\ &+\left(n_\mu\frac{\partial}{\partial x_\nu}+n_\nu\frac{\partial}{\partial x_\mu}\right)n_\lambda\frac{\partial}{\partial x_\lambda}\bigg) \\ &\times\mathfrak{D}^{(1)}(x-x')\epsilon(x-x')d\omega',\end{aligned} \tag{3.43}$$

and

$$\begin{aligned}&\frac{1}{2}\int[j_\mu(x),\, j_\nu(x')]_1\left(\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}+n_\mu\frac{\partial}{\partial x_\nu}n_\lambda\frac{\partial}{\partial x_\lambda}\right) \\ &\qquad\times\mathfrak{D}^{(1)}(x-x')\epsilon(x-x')d\omega' \\ &=-\frac{1}{2}\int\frac{\partial}{\partial x_\nu'}\bigg\{[j_\mu(x),j_\nu(x')]_1\left(\frac{\partial}{\partial x_\mu}+n_\mu n_\lambda\frac{\partial}{\partial x_\lambda}\right) \\ &\qquad\times\mathfrak{D}^{(1)}(x-x')\bigg\}\epsilon(x-x')d\omega' \\ &=-\int_\sigma\left(\frac{\partial}{\partial x_\mu}+n_\mu n_\lambda\frac{\partial}{\partial x_\lambda}\right)\mathfrak{D}^{(1)}(x-x') \\ &\qquad\times[j_\mu(x),\, j_\nu(x')]_1 d\sigma_\nu'=0,\end{aligned}$$

while

$$\begin{aligned}&\frac{1}{2}\int[j_\mu(x),\, j_\nu(x')]_1\left(n_\nu\frac{\partial}{\partial x_\mu}n_\lambda\frac{\partial}{\partial x_\lambda}\right) \\ &\qquad\times\mathfrak{D}^{(1)}(x-x')\epsilon(x-x')d\omega' \\ &=\frac{\partial}{\partial x_\mu}\bigg[\frac{1}{2}\int[j_\mu(x),\, j_\nu(x')]_1 n_\nu n_\lambda\frac{\partial}{\partial x_\lambda} \\ &\qquad\times\mathfrak{D}^{(1)}(x-x')\epsilon(x-x')d\omega'\bigg],\end{aligned} \tag{3.44}$$

since

$$\begin{aligned}&\frac{1}{2}\int[j_\mu(x),\, j_\nu(x')]_1 n_\lambda\frac{\partial}{\partial x_\lambda} \\ &\qquad\times\mathfrak{D}^{(1)}(x-x')n_\nu\frac{\partial}{\partial x_\mu'}\epsilon(x-x')d\omega' \\ &=\frac{1}{2}\int[j_\mu(x),\, j_\nu(x')]_1 n_\lambda\frac{\partial}{\partial x_\lambda} \\ &\qquad\times\mathfrak{D}^{(1)}(x-x')n_\mu\frac{\partial}{\partial x_\nu'}\epsilon(x-x')d\omega' \\ &=-\int_\sigma n_\lambda\frac{\partial}{\partial x_\lambda}\mathfrak{D}^{(1)}(x-x')n_\mu[j_\mu(x),j_\nu(x')]_1 d\sigma_\nu' \\ &=0.\end{aligned} \tag{3.45}$$

Therefore,

$$\begin{aligned}&-\frac{1}{4c}\{(\delta j_\mu(x))_1,\, \mathcal{A}_\mu(x)\}_0 \\ &=-\frac{1}{4c}\{(\delta j_\mu(x))_1,\, A_\mu(x)\}_0 \\ &\quad+\frac{\partial}{\partial x_\mu}\bigg[\frac{i}{8c^2}\int[j_\mu(x),\, j_\nu(x')]_1 n_\nu n_\lambda\frac{\partial}{\partial x_\lambda} \\ &\qquad\times\mathfrak{D}^{(1)}(x-x')\epsilon(x-x')d\omega'\bigg],\end{aligned} \tag{3.46}$$

which establishes the stated equivalence.

The elimination of the first-order terms in (3.1) and (3.2) has thus resulted in the following equations for the new state vectors:

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}=\bigg[-\frac{1}{4c}\{j_\mu(x),\,\delta A_\mu(x)\}-\frac{1}{4c}\{(\delta j_\mu(x))_1,\,A_\mu(x)\}_0-\frac{1}{4c}\{(\delta j_\mu(x))_1,\,A_\mu(x)\}_1\bigg]\Psi[\sigma],\quad(3.47\text{a})$$

$$\left[\frac{\partial A_\mu(x)}{\partial x_\mu}-\frac{1}{2c}\int_\sigma D(x-x')(\delta j_\mu(x'))_1 d\sigma_\mu'\right]\times\Psi[\sigma]=0,\quad(3.47\text{b})$$

and

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}=\bigg[-\frac{1}{4c}\{j_\mu(x),\,\delta A_\mu(x)\}-\frac{1}{4c}\{(\delta j_\mu(x))_1,\,A_\mu(x)\}_0-\frac{1}{4c}\{(\delta j_\mu(x))_1,\,\mathfrak{A}_\mu(x)\}_1\bigg]\Psi[\sigma].\quad(3.48)$$

The matter interaction terms in (3.47a) have been obtained in a natural and direct manner, while the same quantities in (3.48) have resulted from rather elaborate operations designed to unite the longitudinal and transverse field contributions. On the other hand, the elimination of the longitudinal fields has yet to be performed in (3.47a), while the term describing radiation processes in (3.48) requires no further manipulation. To complete the picture of these alternative procedures for dealing with second-order effects, we shall carry out the elimination of the longitudinal field in (3.47a), thus finally performing the process that has already been incorporated in (3.2) and its successor, (3.48).

We first observe that

$$-\frac{1}{4c}\{(\delta j_\mu(x))_1,\,A_\mu(x)\}=-\frac{1}{4c}\{(\delta j_\mu(x))_1,\,\mathfrak{A}_\mu(x)\}-\frac{1}{4c}n_\mu\left\{(\delta j_\mu(x))_1,\,n_\nu\frac{\partial}{\partial x_\nu}(\Lambda(x)-\Lambda'(x))\right\}+\frac{\partial}{\partial x_\mu}\frac{1}{4c}\{(\delta j_\mu(x))_1,\,\Lambda'(x)\},\quad(3.49)$$

of which the last term can be eliminated by the canonical transformation

$$\Psi[\sigma]\rightarrow e^{-iG'[\sigma]}\Psi[\sigma],\qquad G'[\sigma]=\frac{1}{4\hbar c^2}\int_\sigma\{(\delta j_\mu(x))_1,\,\Lambda'(x)\}d\sigma_\mu\quad(3.50)$$

in analogy with (I, 3.19, 3.20). However, the elimination is only to the required second order since the commutation properties of $\delta j_\mu(x)$ are unlike those of $j_\mu(x)$ in the situation cited. The new supplementary condition, correct to the second order, is

$$\bigg[\Lambda(x)-\Lambda'(x)+i[G'[\sigma],\,(\Lambda(x)-\Lambda'(x))]-\frac{1}{2c}\int_\sigma\mathfrak{D}(x-x')(\delta j_\mu(x'))_1 d\sigma_\mu'\bigg]\Psi[\sigma]=0.\quad(3.51)$$

Happily, $\delta j_\mu(x)$ does commute with $\Lambda(x)-\Lambda'(x)$, which is all that is required in order that the new supplementary condition be simply:

$$[\Lambda(x)-\Lambda'(x)]\Psi[\sigma]=0,\quad(3.52)$$

as in (I, 3.26). To verify the stated commutation law, note that

$$\delta j_\mu(x)=\frac{i}{2\hbar c^2}\int[j_\mu(x),\,j_\nu(x')]\epsilon(x-x')\times\left\{\mathfrak{A}_\nu(x')+n_\nu n_\lambda\frac{\partial}{\partial x_\lambda'}(\Lambda(x')-\Lambda'(x'))\right\},\quad(3.53)$$

in which the omitted $\partial\Lambda'(x')/\partial x_\nu'$ term is easily shown to give no contribution. The proof is completed by remarking that there is a vanishing commutator for $\Lambda(x)-\Lambda'(x)$ at two different points. The new form of the supplementary condition ensures that the second term on the right side of (3.49) does not contribute to the state vector equation of motion. Furthermore, the supplementary condition reduces the expression for the current induced by $A_\mu(x)$, (3.53), to that induced by $\mathfrak{A}_\mu(x)$. We have thereby demonstrated that the elimination of the longitudinal fields in (3.47a) yields (3.48).

The coupling of the matter field with itself includes both the interaction of different particles and the self-action of individual particles. Our next task is the separation of the matter interaction terms into these component parts. The basis for such a decomposition in to what may be called one particle and two particle terms lies in the interpretation of the spinors describing the matter field as particle creation and annihilation operators. The operator commutation law

$$[\psi(x), Q]=ie\int_\sigma [\psi(x), (\bar{\psi}(x')\gamma_\mu)_\alpha]\psi_\alpha(x')d\sigma_\mu' \tag{3.54}$$
$$=e\psi(x),$$

when applied to an eigenstate of total charge, $\Psi(Q')$, states that

$$Q\psi(x)\Psi(Q')=(Q'-e)\psi(x)\Psi(Q'). \tag{3.55}$$

Evidently $\psi(x)$ acts as an operator decreasing the charge of the system by e and therefore either annihilates a particle of charge e, or creates a particle of charge $-e$. Similarly, $\psi'(x)$ or $\bar{\psi}(x)$ either creates a particle of charge e or destroys a particle of charge $-e$. Quantities of the typical form $\bar{\psi}\psi$ consequently induce such effects as the annihilation of a particle in one state and the creation of a similar particle in another state, which can be viewed as the transition of particle between the two states. Such quantities may be called one particle operators, although the nomenclature is only strictly accurate when the vacuum expectation value of $\bar{\psi}\psi$ is subtracted. The latter arises from the creation and subsequent annihilation of particles of charge $-e$, occurring in the absence of matter as a vacuum fluctuation. The more complicated operators of the type $\bar{\psi}\psi\bar{\psi}\psi$ induce a variety of effects including the annihilation of two particles and creation of two others in different states, which is to be regarded as the transition of a pair of particles from one set of states to another. Such two-particle effects are to be distinguished from phenomena in which one particle makes a transition while another is created and then destroyed. This one-particle transition coupled with a vacuum fluctuation is observationally indistinguishable from the simple one particle effects previously mentioned. Of course, $\bar{\psi}\psi\bar{\psi}\psi$ also produces phenomena in the vacuum, in which both transitions are vacuum fluctuations. It is in this way, by successively restricting the possible transitions to be vacuum fluctuations, that $\bar{\psi}\psi\bar{\psi}\psi$, and more general expressions, can be decomposed into operators associated with various numbers of particles.

The operator to which we shall apply this decomposition is

$$\{j_\mu(x), j_\nu(x')\}=-e^2c^2\{\tfrac{1}{2}[\bar{\psi}(x)\gamma_\mu, \psi(x)],$$
$$\times\tfrac{1}{2}[\bar{\psi}(x')\gamma_\nu, \psi(x')]\}. \tag{3.56}$$

In order to evaluate its vacuum expectation value, it is merely necessary to replace the bilinear products $\bar{\psi}\psi$ and $\psi\bar{\psi}$, in all possible combinations, by their vacuum expectation values. However, since $j_\mu(x)$ and $j_\nu(x')$ have a vanishing vacuum expectation value, it is only products of the type $\bar{\psi}(x)\psi(x')$, $\bar{\psi}(x')\psi(x)$, $\psi(x)\bar{\psi}(x')$ and $\psi(x')\bar{\psi}(x)$ that need be included. To discuss one term in detail, consider

$$(\gamma_\mu)_{\alpha\beta}(\gamma_\nu)_{\gamma\delta}\bar{\psi}_\alpha(x)\psi_\beta(x)\bar{\psi}_\gamma(x')\psi_\delta(x'). \tag{3.57}$$

The operator $\psi_\delta(x')$, acting on the vacuum state vector, produces a particle of charge $-e$ (an electron, say). The effect of $\bar{\psi}_\gamma(x')$ can be to immediately annihilate this particle, but, for the reason mentioned above, such a term would be cancelled on considering the second part of the expression for $j_\nu(x')$. Thus, the essential result produced by $\bar{\psi}_\gamma(x')$ will be the creation of a particle with charge e (positron). The remaining two operators must destroy the electron and positron that have been created in order that we deal with a vacuum effect. Hence $\psi_\beta(x)$ must annihilate the particle generated by $\bar{\psi}_\gamma(x')$, which effectively replaces $\psi_\beta(x)\bar{\psi}_\gamma(x')$ by its vacuum expectation value. Finally, $\bar{\psi}_\alpha(x)$ must destroy the electron created by $\psi_\delta(x')$, thus replacing the product $\bar{\psi}_\alpha(x)\psi_\delta(x')$ by its vacuum expectation value. There is no difficulty in associating $\bar{\psi}_\alpha(x)$ with $\psi_\delta(x')$, despite the two intervening operators, since $\bar{\psi}_\alpha(x)$ effectively anticommutes with both operators, acting as it does on a different particle from those affected by the operator product $\psi_\beta(x)\bar{\psi}_\gamma(x')$. Therefore, the vacuum expectation value of (3.57) is

$$(\gamma_\mu)_{\alpha\beta}(\gamma_\nu)_{\gamma\delta}\langle\bar{\psi}_\alpha(x)\psi_\delta(x')\rangle_0\langle\psi_\beta(x)\bar{\psi}_\gamma(x')\rangle_0. \tag{3.58}$$

But

$$\begin{aligned}\langle\psi_\beta(x)\bar{\psi}_\gamma(x')\rangle_0 &= \tfrac{1}{2}\{\psi_\beta(x),\bar{\psi}_\gamma(x')\} \\ &\quad+\tfrac{1}{2}\langle[\psi_\beta(x),\bar{\psi}_\gamma(x')]\rangle_0 \\ &= -\frac{i}{2}S_{\beta\gamma}(x-x')-\tfrac{1}{2}S_{\beta\gamma}{}^{(1)}(x-x') \\ &= -iS_{\beta\gamma}{}^{(+)}(x-x'),\end{aligned}\tag{3.59}$$

and

$$\begin{aligned}\langle\bar{\psi}_\alpha(x)\psi_\delta(x')\rangle_0 &= -iS_{\delta\alpha}(x'-x)-\langle\psi_\delta(x'),\bar{\psi}_\alpha(x)\rangle_0 \\ &= -iS_{\delta\alpha}(x'-x)+iS_{\delta\alpha}{}^{(+)}(x'-x) \\ &= -iS_{\delta\alpha}{}^{(-)}(x'-x),\end{aligned}\tag{3.60}$$

whence (3.58) becomes

$$-Tr[\gamma_\mu S^{(+)}(x-x')\gamma_\nu S^{(-)}(x'-x)].\tag{3.61}$$

Continuing in this manner, we find that

$$\begin{aligned}&\{j_\mu(x),j_\nu(x')\}_0 \\ &= e^2c^2Tr[\gamma_\mu S^{(+)}(x-x')\gamma_\nu S^{(-)}(x'-x) \\ &\qquad+\gamma_\mu S^{(-)}(x-x')\gamma_\nu S^{(+)}(x'-x)] \\ &= \frac{e^2c^2}{2}Tr[\gamma_\mu S(x-x')\gamma_\nu S(x'-x) \\ &\qquad+\gamma_\mu S^{(1)}(x-x')\gamma_\nu S^{(1)}(x'-x)].\end{aligned}\tag{3.62}$$

An evaluation of this trace, employing the same method used to derive (2.15), yields

$$\begin{aligned}&\{j_\mu(x),j_\nu(x')\}_0 \\ &= 2e^2c^2\Bigg\{2\frac{\partial\Delta(x-x')}{\partial x_\mu}\frac{\partial\Delta(x-x')}{\partial x_\nu} \\ &\quad-\delta_{\mu\nu}\left[\left(\frac{\partial\Delta(x-x')}{\partial x_\lambda}\right)^2+\kappa_0{}^2(\Delta(x-x'))^2\right] \\ &\quad-2\frac{\partial\Delta^{(1)}(x-x')}{\partial x_\mu}\frac{\partial\Delta^{(1)}(x-x')}{\partial x_\nu} \\ &\quad+\delta_{\mu\nu}\left[\left(\frac{\partial\Delta^{(1)}(x-x')}{\partial x_\lambda}\right)^2 \right. \\ &\qquad\left.+\kappa_0{}^2(\Delta^{(1)}(x-x'))^2\right]\Bigg\}.\end{aligned}\tag{3.63}$$

This, in turn, can be replaced by a form analogous to (2.23):

$$\{j_\mu(x),j_\nu(x')\}_0 = 4e^2c^2\left(\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}-\delta_{\mu\nu}\square^2\right)\times(\bar{L}(\lambda)-L^{(1)}(\lambda)),\tag{3.64}$$

where

$$\begin{aligned}4\left(\frac{\partial\bar{\Delta}(\lambda)}{\partial\lambda}\right)^2 &= \frac{\partial^2\bar{L}(\lambda)}{\partial\lambda^2}, \\ \left(\frac{\partial\Delta^{(1)}(\lambda)}{\partial\lambda}\right)^2 &= \frac{\partial^2L^{(1)}(\lambda)}{\partial\lambda^2}.\end{aligned}\tag{3.65}$$

In particular,

$$\{j_\mu(x),j_\mu(x')\}_0 = -12e^2c^2\square^2(\bar{L}(\lambda)-L^{(1)}(\lambda)).\tag{3.66}$$

To find the one particle component of (3.56), we shall isolate that part of the operator that induces the transition of a positron, say, from one state to another with no other observable change in the matter field. We may again consider the typical term (3.57). The operator $\psi_\delta(x')$ either annihilates the single positron present or creates an electron. If it is the destruction of the original positron that occurs, the second operator $\bar{\psi}_\gamma(x')$ can only re-create a positron, in what is generally another state. The third operator can annihilate the previously generated positron, thus forming the vacuum expectation value of $\psi_\beta(x)\bar{\psi}_\gamma(x')$, or create an electron. However, if the latter occurs, $\bar{\psi}_\alpha(x)$ must annihilate that electron, in order that only a one-particle transition occur. But this would form the vacuum expectation value of $\bar{\psi}_\alpha(x)\psi_\beta(x)$, which, as we know, is effectively cancelled by the second term of $j_\mu(x)$. Thus, if the original positron is first annihilated, the only event that can ensue is a vacuum fluctuation, followed by the creation of the positron in its final state. If, on the other hand, the first process to occur is the creation of an electron, this must be followed by the creation of a positron in what will turn out to be the final state; the immediate annihilation of the electron need not be considered, for previously stated reasons. The third operator $\psi_\beta(x)$ can now only annihilate the original positron, and $\bar{\psi}_\alpha(x)$ destroys the electron. The two sets of transitions that have thus been described in detail are those induced by the operator

$$(\gamma_\mu)_{\alpha\beta}(\gamma_\nu)_{\gamma\delta}\{[\bar{\psi}_\alpha(x)\psi_\delta(x')]_1\langle\psi_\beta(x)\bar{\psi}_\gamma(x')\rangle_0 + [\psi_\beta(x)\bar{\psi}_\gamma(x')]_1\langle\bar{\psi}_\alpha(x)\psi_\delta(x')\rangle_0\}, \quad (3.67)$$

which, indeed, is the one-particle part of (3.57) since it has a vanishing vacuum expectation value.

It is now only a short step to the one-particle part of (3.56):

$$\{j_\mu(x), j_\nu(x')\}_1 = -\frac{e^2c^2}{2}(\gamma_\mu)_{\alpha\beta}(\gamma_\nu)_{\gamma\delta}\{[\bar{\psi}_\alpha(x), \psi_\delta(x')]_1 \times\langle[\psi_\beta(x), \bar{\psi}_\gamma(x')]\rangle_0 + [\psi_\beta(x), \bar{\psi}_\gamma(x')]_1 \times\langle[\bar{\psi}_\alpha(x), \psi_\delta(x')]\rangle_0\}$$
$$= \frac{e^2c^2}{2}\{[\bar{\psi}(x), \gamma_\mu S^{(1)}(x-x')\gamma_\nu\psi(x')]_1 + [\bar{\psi}(x')\gamma_\nu S^{(1)}(x'-x)\gamma_\mu, \psi(x)]_1\}, \quad (3.68)$$

and the actual quantity of interest

$$\{j_\mu(x), j_\mu(x')\}_1 = \frac{e^2c^2}{2}\{[\bar{\psi}(x), \gamma_\mu S^{(1)}(x-x')\gamma_\mu\psi(x')]_1 + [\bar{\psi}(x')\gamma_\mu S^{(1)}(x'-x)\gamma_\mu, \psi(x)]_1\}. \quad (3.69)$$

Of course, the two particle part of (3.56) is simply

$$\{j_\mu(x), j_\nu(x')\}_2 = \{j_\mu(x), j_\nu(x')\} - \{j_\mu(x), j_\nu(x')\}_1 - \{j_\mu(x), j_\nu(x')\}_0, \quad (3.70)$$

and we shall have no occasion to further simplify it.

It is now possible to write the second-order state vector equation in the detailed form

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(x)} = \{\mathcal{H}_{0,0} + \mathcal{H}_{1,0}(x) + \mathcal{H}_{2,0}(x) + \mathcal{H}_{1,1}(x)\}\Psi[\sigma], \quad (3.71)$$

where

$$\mathcal{H}_{0,0} = -\frac{1}{4c}\{j_\mu(x), \delta A_\mu(x)\}_0 = -\frac{1}{4c^2}\int\{j_\mu(x), j_\mu(x')\}_0 \times\bar{D}(x-x')d\omega', \quad (3.72)$$

$$\mathcal{H}_{1,0}(x) = -\frac{1}{4c}\{j_\mu(x), \delta A_\mu(x)\}_1 - \frac{1}{4c}\{(\delta j_\mu(x))_1, A_\mu(x)\}_0$$
$$= -\frac{1}{4c^2}\int\{j_\mu(x), j_\mu(x')\}_1\bar{D}(x-x')d\omega' - \frac{i}{8c^2}\int[j_\mu(x), j_\mu(x')]_1 \times\epsilon(x-x')D^{(1)}(x-x')d\omega', \quad (3.73)$$

$$\mathcal{H}_{2,0}(x) = -\frac{1}{4c}\{j_\mu(x), \delta A_\mu(x)\}_2 = -\frac{1}{4c^2}\int\{j_\mu(x), j_\mu(x')\}_2 \times\bar{D}(x-x')d\omega', \quad (3.74)$$

and

$$\mathcal{H}_{1,1}(x) = -\frac{1}{4c}\{(\delta j_\mu(x))_1, \mathcal{A}_\mu(x)\}_1 = -\frac{i}{8\hbar c^3}\int[j_\mu(x), j_\nu(x')]_1\epsilon(x-x') \times\{\mathcal{A}_\mu(x), \mathcal{A}_\mu(x')\}_1 d\omega'. \quad (3.75)$$

In these quantities the subscripts refer to the number of particles and light quanta whose transitions are described by the term in question. The change in the properties of the vacuum arising from the coupling between matter and radiation is produced by

$$\mathcal{H}_{0,0} = 3e^2\int\Box'^2(\bar{L}(\lambda) - L^{(1)}(\lambda))\bar{D}(x-x')d\omega' = -3e^2(\bar{L}(0) - L^{(1)}(0)), \quad (3.76)$$

which implies nothing of physical interest, however. We turn, at last, to the quantity describing the altered properties of individual particles:

$$\mathcal{H}_{1,0}(x) = -\frac{e^2}{8}\int\{[\bar{\psi}(x), \gamma_\mu S^{(1)}(x-x')\gamma_\mu\psi(x')]_1\bar{D}(x-x') + [\bar{\psi}(x), \gamma_\mu\bar{S}(x-x')\gamma_\mu\psi(x')]_1 D^{(1)}(x-x') + [\bar{\psi}(x')\gamma_\mu S^{(1)}(x'-x)\gamma_\mu, \psi(x)]_1\bar{D}(x-x') + [\bar{\psi}(x')\gamma_\mu\bar{S}(x'-x)\gamma_\mu, \psi(x)]_1 D^{(1)}(x-x')\}d\omega'$$
$$= \tfrac{1}{4}[\bar{\psi}(x), \phi(x)]_1 + \tfrac{1}{4}[\bar{\phi}(x), \psi(x)]_1, \quad (3.77)$$

where

$$\phi(x)=-\frac{e^2}{2}\int\gamma_\mu[\bar{D}(x-x')S^{(1)}(x-x')+D^{(1)}(x-x')\bar{S}(x-x')]\gamma_\mu\psi(x')d\omega'. \quad (3.78)$$

We shall prove that $\phi(x)$ is simply a multiple of $\psi(x)$,

$$\phi(x)=\delta mc^2\psi(x), \quad (3.79)$$

whence

$$\mathcal{H}_{1,0}(x)=\delta mc^2\tfrac{1}{2}[\bar{\psi}(x),\psi(x)]_1=\delta mc^2\tfrac{1}{2}[\bar{\psi}(x)\psi(x)+\bar{\psi}'(x)\psi'(x)]_1. \quad (3.80)$$

It will be evident that δm is the electromagnetic mass of the electron.

The identity

$$\gamma_\mu\left(\gamma_\lambda\frac{\partial}{\partial x_\lambda}-\kappa_0\right)\gamma_\mu=-2\left(\gamma_\lambda\frac{\partial}{\partial x_\lambda}+2\kappa_0\right) \quad (3.81)$$

enables $\phi(x)$ to be written:

$$\phi(x)=e^2\int\left\{\gamma_\lambda\left[\bar{D}(x-x')\frac{\partial}{\partial x_\lambda}\Delta^{(1)}(x-x')+D^{(1)}(x-x')\frac{\partial}{\partial x_\lambda}\bar{\Delta}(x-x')\right]+2\kappa_0[\bar{D}(x-x')\Delta^{(1)}(x-x')+D^{(1)}(x-x')\bar{\Delta}(x-x')]\right\}\psi(x')d\omega'. \quad (3.82)$$

We now define $P(\lambda)$, a function of $\lambda=-(x_\mu-x_\mu')^2$ according to

$$\frac{\partial P(\lambda)}{\partial\lambda}=\bar{D}(\lambda)\frac{\partial\Delta^{(1)}(\lambda)}{\partial\lambda}+D^{(1)}(\lambda)\frac{\partial\bar{\Delta}(\lambda)}{\partial\lambda}. \quad (3.83)$$

The utility of this quantity stems from the relation

$$\bar{D}(x)\frac{\partial\Delta^{(1)}(x)}{\partial x_\mu}+D^{(1)}(x)\frac{\partial\bar{\Delta}(x)}{\partial x_\mu}=-2x_\mu\frac{\partial P(\lambda)}{\partial\lambda}=\frac{\partial P(\lambda)}{\partial x_\mu}, \quad (3.84)$$

which permits the first term of (3.82) to be simplified,

$$e^2\int\gamma_\mu\frac{\partial P(\lambda)}{\partial x_\mu}\psi(x')d\omega'=e^2\int P(\lambda)\gamma_\mu\frac{\partial}{\partial x_\mu'}\psi(x')d\omega'=-e^2\kappa_0\int P(\lambda)\psi(x')d\omega'. \quad (3.85)$$

Hence,

$$\phi(x)=e^2\kappa_0\int Q(\lambda)\psi(x')d\omega', \quad (3.86)$$

where

$$Q(\lambda)=2[\bar{D}(\lambda)\Delta^{(1)}(\lambda)+D^{(1)}(\lambda)\bar{\Delta}(\lambda)]-P(\lambda). \quad (3.87)$$

The explicit construction of $P(\lambda)$ and $Q(\lambda)$ proceeds analogously to that of $G(\lambda)$ in the second section. We first note that

$$\frac{\partial P(\lambda)}{\partial\lambda}=-\frac{1}{(2\pi)^4}\int\exp\left(i\lambda(\alpha+\beta)+i\frac{\kappa_0^2}{4\alpha}\right)\times\alpha\frac{1}{2}\left(\frac{\alpha}{|\alpha|}+\frac{\beta}{|\beta|}\right)d\alpha d\beta,$$

and

$$P(\lambda)=\frac{i}{(2\pi)^4}\int\exp\left(i\lambda(\alpha+\beta)+i\frac{\kappa_0^2}{4\alpha}\right)\times\frac{\alpha}{\alpha+\beta}\frac{1}{2}\left(\frac{\alpha}{|\alpha|}+\frac{\beta}{|\beta|}\right)d\alpha d\beta. \quad (3.38)$$

Hence,

$$Q(\lambda)=\frac{i}{(2\pi)^4}\int\exp\left(i\lambda(\alpha+\beta)+i\frac{\kappa_0^2}{4\alpha}\right)\times\left(2-\frac{\alpha}{\alpha+\beta}\right)\frac{1}{2}\left(\frac{\alpha}{|\alpha|}+\frac{\beta}{|\beta|}\right)d\alpha d\beta. \quad (3.89)$$

In terms of the variables v and w, defined by (2.34), this reads

$$Q=\frac{i}{4(2\pi)^4}\kappa_0^4\int_{-1}^{1}\frac{3-v}{(1-v^2)^2}dv\int_{-\infty}^{\infty}\frac{dw}{w^3}\times\exp\left[iw\frac{1-v}{2}+i\frac{\lambda\kappa_0^2}{w(1-v^2)}\right], \quad (3.90)$$

which in turn becomes

$$Q=\frac{1}{16(2\pi)^6}\int(dk)\exp(ik_\mu(x_\mu-x_\mu'))\times\int_{-1}^{1}(3-v)dv\int_{-\infty}^{\infty}\frac{dw}{|w|}\times\exp\left(i\left[\frac{1-v}{2}+\frac{k_\mu^2}{4\kappa_0^2}(1-v^2)\right]w\right),\quad(3.91)$$

on using the integral representation (2.36).

We now remark that $\psi(x)$ satisfies the second order differential equation

$$(\Box^2-\kappa_0^2)\psi(x)=0,\quad(3.92)$$

which implies that a Fourier decomposition of $\psi(x)$ into plane waves of the form $e^{ik_\mu x_\mu}$ involves only such propagation vectors that obey

$$k_\mu^2=-\kappa_0^2.\quad(3.93)$$

Therefore, in evaluating the integral

$$\int Q(\lambda)\psi(x')d\omega',$$

with $Q(\lambda)$ expressed as a Fourier integral involving $e^{ik_\mu(x_\mu-x_\mu')}$, multiplied by a function of k_μ^2:

$$Q(\lambda)=\frac{1}{(2\pi)^4}\int\exp(ik_\mu(x_\mu-x_\mu'))F(k_\mu^2)(dk),\quad(3.94)$$

the latter quantity may be replaced by $-\kappa_0^2$. Thus

$$\int Q(\lambda)\psi(x')d\omega'=\frac{1}{(2\pi)^4}\int(dk)\int d\omega'\times\exp(ik_\mu(x_\mu-x_\mu'))F(-\kappa_0^2)\psi(x')=F(-\kappa_0^2)\psi(x),\quad(3.95)$$

which confirms the statement that $\phi(x)$ is proportioned to $\psi(x)$, and yields as the value of the constant:

$$\delta mc^2=\frac{\alpha}{8\pi}m_0c^2\int_{-1}^{1}(3-v)dv\times\int_0^\infty\frac{dw}{w}\cos\left(\frac{1-v}{2}\right)^2w.\quad(3.96)$$

To complete the evaluation, it is convenient to integrate by parts, according to

$$\frac{\delta m}{m_0}=\frac{\alpha}{16\pi}\int_{-1}^{1}d[(v-5)(1-v)]\times\int_0^\infty\frac{dw}{w}\cos\left(\frac{1-v}{2}\right)^2w$$
$$=\frac{\alpha}{8\pi}\left[6\int_0^\infty\frac{\cos w}{w}dw+\int_{-1}^{1}(5-v)\left(\frac{1-v}{2}\right)^2dv\times\int_0^\infty\sin\left(\frac{1-v}{2}\right)^2wdw\right]$$
$$=\frac{3\alpha}{2\pi}\left[\frac{1}{2}\log\frac{1}{\gamma w_0}+\frac{5}{6}\right],\quad(3.97)$$

whereby we obtain a logarithmically divergent result for the electromagnetic mass of the electron or positron. An alternative evaluation can be given, which permits comparison with previous treatments,[7] by employing directly the Fourier integral representations for the functions $D(x)$, $\Delta(x)$, $D^{(1)}(x)$ and $\Delta^{(1)}(x)$, in (3.82). One thus obtains the electromagnetic mass as an integral over the momenta of the virtual quanta involved in the self energy process, with the result

$$\frac{\delta m}{m_0}=\frac{3\alpha}{2\pi}\left[\log\frac{K+K_0}{\kappa_0}-\frac{1}{6}\right]_{K=\infty},\quad(3.98)$$

where $K_0=(K^2+\kappa_0^2)^{\frac{1}{2}}$. Evidently $1/w_0\sim(K/\kappa_0)^2$.

To justify the identification of δm with the electromagnetic mass, we must show that it is possible to remove the term $\mathcal{H}_{1,0}(x)$ from (3.71) and thereby alter the equation of motion for the matter field into that of a particle of mass $m=m_0+\delta m$, thus demonstrating the unity of the two contributions to the actual electron mass. Accordingly, we introduce the state vector transformation

$$\Psi[\sigma]=U[\sigma]\Psi'[\sigma],\quad(3.99)$$

where $U[\sigma]$ is designed to remove the variation associated with $\mathcal{H}_{1,0}(x)$ from $\Psi'[\sigma]$ and therefore

[7] V. Weisskopf. Zeits. f. Physik **89**, 27 (1934); **90**, 817 (1934), and Phys. Rev. **56**, 72 (1939).

is subject to the equation of motion

$$i\hbar c\frac{\delta U[\sigma]}{\delta\sigma(x)}=\mathcal{H}_{1,0}(x)U[\sigma]. \quad (3.100)$$

Thus, we return to a Heisenberg representation for the description of self-energy effects. The transformation (3.99) induces a concomitant change in the matter field operators,

$$\boldsymbol{\psi}(x)=U^{-1}[\sigma]\psi(x)U[\sigma], \quad (3.101)$$

in which we use bold face letters to designate the new operators as well as the new state vector. To construct the equation of motion satisfied by $\boldsymbol{\psi}(x)$, we require an equation analogous to (I, 2.9). Since the roles of interaction and Heisenberg representations have been interchanged, we find that

$$\frac{\partial\boldsymbol{\psi}(x)}{\partial x_\mu}=U^{-1}[\sigma]\frac{\partial\psi(x)}{\partial x_\mu}U[\sigma] -\frac{i}{\hbar c}\int_\sigma[\boldsymbol{\psi}(x'),\boldsymbol{H}_{1,0}(x)]d\sigma_\mu', \quad (3.102)$$

whence

$$\begin{aligned}\left(\gamma_\mu\frac{\partial}{\partial x_\mu}+\kappa_0\right)\boldsymbol{\psi}(x) &=-\frac{i}{\hbar c}\int_\sigma[\gamma_\mu\boldsymbol{\psi}(x'),\boldsymbol{H}_{1,0}(x)]d\sigma_\mu' \\ &=-\frac{\delta mc}{\hbar}i\int_\sigma[\gamma_\mu\boldsymbol{\psi}(x'),\bar{\boldsymbol{\psi}}_\alpha(x)]d\sigma_\mu'\boldsymbol{\psi}_\alpha(x) \\ &=-\delta\kappa\boldsymbol{\psi}(x),\end{aligned} \quad (3.103)$$

where

$$\delta\kappa=\delta mc/\hbar. \quad (3.104)$$

Finally, then

$$\left(\gamma_\mu\frac{\partial}{\partial x_\mu}+\kappa\right)\boldsymbol{\psi}(x)=0 \quad (3.105)$$

with

$$\kappa=mc/\hbar,\quad m=m_0+\delta m, \quad (3.106)$$

which is the desired result. The σ-variation of the state vector that remains after removal of $\mathcal{H}_{1,0}(x)$, and $\mathcal{H}_{0,0}$ (which we may include, without altering the previous considerations) is described, correct to the second order, by the equation of motion

$$i\hbar c\frac{\delta\boldsymbol{\Psi}[\sigma]}{\delta\sigma(x)}=\{\boldsymbol{H}_{2,0}(x)+\boldsymbol{H}_{1,1}(x)\}\boldsymbol{\Psi}[\sigma], \quad (3.107)$$

which governs the second order interaction of a particle with a light quantum or another particle.

It will be our final task to examine the form which the energy-momentum quantities assume as a result of the succession of transformations which have culminated in (3.107). In particular, we shall again confirm the complete amalgamation of the mechanical and electromagnetic mass of the electron. We may first remark on the generality of the arguments that led to the expression for the energy-momentum four-vector in the interaction representation (I, 2.52):

$$P_\mu[\sigma]=P_\mu^{(0)}-\frac{1}{c}\int_\sigma\mathcal{H}(x)d\sigma_\mu \quad (3.108a)$$

$$\mathcal{H}(x)=-\frac{1}{c}j_\mu(x)A_\mu(x). \quad (3.108b)$$

This form must be preserved on subjecting the state vector to a transformation, provided one employs the appropriately transformed $\mathcal{H}(x)$. We shall prove this by direct calculation, however. The energy-momentum operator associated with the new state vector in the transformation (3.7) is, to second order,

$$\begin{aligned}P_\mu[\sigma]&=e^{iS[\sigma]}\left[P_\mu^{(0)}+\frac{1}{c^2}\int_\sigma j_\nu(x)A_\nu(x)d\sigma_\mu\right]e^{-iS[\sigma]} \\ &=P_\mu^{(0)}+i[S[\sigma],P_\mu^{(0)}] \\ &\quad-\tfrac{1}{2}[S[\sigma],[S[\sigma],P_\mu^{(0)}]] \\ &\quad+\frac{1}{c^2}\int_\sigma j_\nu(x)A_\nu(x)d\sigma_\mu \\ &\quad+i\left[S[\sigma],\frac{1}{c^2}\int_\sigma j_\nu(x)A_\nu(x)d\sigma_\mu\right].\end{aligned} \quad (3.109)$$

The interpretation of $P_\mu^{(0)}$ as the displacement operator for the independent fields, when applied to the displacement of a functional, leads to

$$\frac{i}{\hbar}[S[\sigma],P_\mu^{(0)}]=\int_\sigma\frac{\delta S[\sigma]}{\delta\sigma(x)}d\sigma_\mu, \quad (3.110)$$

whence

$$i[S[\sigma], P_\mu^{(0)}]+\frac{1}{c^2}\int_\sigma j_\nu(x)A_\nu(x)d\sigma_\mu=0, \quad (3.111)$$

and

$$P_\mu[\sigma]=P_\mu^{(0)}-\frac{1}{c}\int_\sigma d\sigma_\mu\frac{i}{2}[j_\nu(x)A_\nu(x), S[\sigma]]. \quad (3.112)$$

This indeed is still of the form (3.108a), with $\mathcal{H}(x)$ the quantity occurring in (3.14). There is no difficulty in proceeding to the form

$$P_\mu[\sigma]=P_\mu^{(0)}-\frac{1}{c}\int_\sigma[\mathcal{H}_{0,0}+\mathcal{H}_{1,0}(x)+\mathcal{H}_{2,0}(x)+\mathcal{H}_{1,1}(x)]d\sigma_\mu, \quad (3.113)$$

associated with the state vector equation (3.71). The last transformation to be considered is (3.99), which provides us with the energy-momentum four vector:

$$\mathbf{P}_\mu[\sigma]=U^{-1}[\sigma]P_\mu^{(0)}U[\sigma]-\frac{1}{c}\int_\sigma[\mathcal{H}_{0,0}+\boldsymbol{H}_{1,0}(x)+\boldsymbol{H}_{2,0}(x)+\boldsymbol{H}_{1,1}(x)]d\sigma_\mu. \quad (3.114)$$

Now, according to (I, 1.64) and (3.102),

$$U^{-1}[\sigma]P_\mu^{(0)}U[\sigma]=\mathbf{P}_\mu^{(0)}+\frac{i}{2c}\int_\sigma d\sigma_\mu \times[\bar{\psi}(x), [\gamma_\mu\psi(x'), \boldsymbol{H}_{1,0}(x)]]_1 d\sigma_\nu', \quad (3.115)$$

in which the subscript one indicates that $P_\mu^{(0)}$, or $\mathbf{P}_\mu^{(0)}$, is constructed to have a vanishing vacuum expectation value. On evaluating the second term of (3.115) as:

$$\frac{1}{c}\int_\sigma d\sigma_\mu i\{\gamma_\mu\psi(x'), \bar{\psi}_\alpha(x)\}\delta mc^2\tfrac{1}{2}[\bar{\psi}(x), \psi_\alpha(x)]_1 d\sigma_\nu'$$
$$=\frac{1}{c}\int_\sigma \delta mc^2\tfrac{1}{2}[\bar{\psi}(x), \psi(x)]d\sigma_\nu=\frac{1}{c}\int_\sigma \boldsymbol{H}_{1,0}(x)d\sigma_\nu,$$

we find

$$\mathbf{P}_\mu[\sigma]=\mathbf{P}_\mu^{(0)}-\frac{1}{c}\int_\sigma[\boldsymbol{H}_{2,0}(x)+\boldsymbol{H}_{1,1}(x)]d\sigma_\mu \quad (3.116)$$

from which the vacuum term $\mathcal{H}_{0,0}$ has been removed, in accordance with the trivial possibility of adding a multiple of $\delta_{\mu\nu}$ to the energy-momentum tensor of the system. With this result, we have confirmed that, for an individual particle, the energy and momentum modifications produced by self-interaction effects are entirely accounted for by the addition of the electromagnetic proper mass δm to the mechanical proper mass m_0—an unobservable mass renormalization.

APPENDIX

In the course of this development of quantum electrodynamics, various functions associated with the electromagnetic and matter fields have been defined, notably $D(x)$, $D^{(1)}(x)$, $\Delta(x)$ and $\Delta^{(1)}(x)$. It is now our task to construct these functions explicitly. We begin with the invariant function associated with the matter field, $\Delta(x)$ (from which $D(x)$ can be obtained by placing $\kappa_0=0$). Its construction is facilitated by considering the associated function

$$\bar{\Delta}(x)=-\tfrac{1}{2}\Delta(x)\epsilon(x)=\tfrac{1}{2}\Delta(x)\frac{\epsilon_\mu x_\mu}{|\epsilon_\mu x_\mu|}, \quad (A.1)$$

where $\epsilon(x)$ is $+1$ or -1 according as x_0 is positive or negative. This sign factor is effectively an invariant since only time-like vectors x_μ need be considered in (A.1). This is emphasized by the invariant representation of $\epsilon(x)$:

$$\epsilon(x)=-\frac{\epsilon_\mu x_\mu}{|\epsilon_\mu x_\mu|} \quad (A.2)$$

where ϵ_μ is an arbitrary time-like vector with $\epsilon_0>0$. It will first be noted that

$$(\Box^2-\kappa_0^2)\bar{\Delta}(x)=0, \quad x_\mu\neq 0 \quad (A.3)$$

since only at the time-like neighborhood of the origin is a sign change of $\epsilon(x)$ combined with a non-vanishing value of $\Delta(x)$. To evaluate the left side of (A.3) at the origin, we consider

$$\operatorname{Lim}\int_{\delta\omega}d\omega(\Box^2-\kappa_0^2)\bar{\Delta}(x)=\operatorname{Lim}\left[\int_{\sigma_+}d\sigma_\mu\frac{\partial\bar{\Delta}(x)}{\partial x_\mu}-\int_{\sigma_-}d\sigma_\mu\frac{\partial\bar{\Delta}(x)}{\partial x_\mu}\right], \quad (A.4)$$

in which the region of integration $\delta\omega$ is extended between two space-like surfaces σ_+ and σ_-, which lie in the future and past, respectively, relative to the origin, and coincide in the limit with the space-like surface σ through the origin. Thus

$$\operatorname{Lim}\int_{\delta\omega}d\omega(\Box^2-\kappa_0^2)\bar{\Delta}(x)=-\int_\sigma\frac{\partial\Delta(x)}{\partial x_\mu}d\sigma_\mu=-1, \quad (A.5)$$

which shows that

$$(\Box^2-\kappa_0^2)\bar{\Delta}(x)=-\delta(x), \quad (A.6)$$

where

$$\delta(x)=\delta(x_0)\delta(x_1)\delta(x_2)\delta(x_3) \quad (A.7)$$

is the four-dimensional delta-function. Evidently $\bar{\Delta}(x)$

plays the role of a four-dimensional Green's function. In terms of the integral representation

$$\delta(x)=\frac{1}{(2\pi)^4}\int \exp(ik_\mu x_\mu)(dk), \qquad (A.8)$$

where

$$(dk)=dk_0dk_1dk_2dk_3, \qquad (A.9)$$

we obtain as a particular solution of (A.6),

$$\bar{\Delta}(x)=\frac{1}{(2\pi)^4}P\int \frac{\exp(ik_\mu x_\mu)}{k_\lambda^2+\kappa_0^2}(dk) \qquad (A.10)$$

which, as will be shown, provides a function $\Delta(x)$ that satisfies the equations of definition (I, 2.18).

The integral representation

$$P\frac{1}{\tau}=-\frac{i}{2}\int_{\infty-}^{\infty} e^{ia\tau}\frac{a}{|a|}da \qquad (A.11)$$

with $\tau=k_\lambda^2+\kappa_0^2$, enables the integration over k space to be effected:

$$\begin{aligned}\bar{\Delta}(x)&=-\frac{i}{2(2\pi)^4}\int_{-\infty}^{\infty}\frac{a}{|a|}da\int (dk)\\ &\quad\times\exp(iak_\mu^2+ik_\mu x_\mu)\exp(ia\kappa_0^2)\\ &=\frac{1}{32\pi^2}\int_{\infty-}^{\infty}\frac{da}{a^2}\exp\left(-i\frac{x_\mu^2}{4a}+ia\kappa_0^2\right) \qquad (A.12)\end{aligned}$$

with the aid of the formula

$$\begin{aligned}\int (dk)\exp(iak_\mu^2+ik_\mu x_\mu)&=\int (dk)\exp(iak_\mu^2)\exp-\left(i\frac{x_\mu^2}{4a}\right)\\ &=i\frac{\pi^2}{a|a|}\exp\left(-i\frac{x_\mu^2}{4a}\right). \qquad (A.13)\end{aligned}$$

The introduction of the new variable,

$$\alpha=1/4a, \qquad (A.14)$$

brings (A.12) into the form

$$\bar{\Delta}(x)=\frac{1}{8\pi^2}\int_{-\infty}^{\infty}\exp\left(i\lambda\alpha+i\frac{\kappa_0^2}{4\alpha}\right)d\alpha, \qquad (A.15)$$

where

$$\lambda=-x_\mu^2. \qquad (A.16)$$

Equivalent forms of (A.15) are:

$$\begin{aligned}\bar{\Delta}(x)=\bar{\Delta}(\lambda)&=\frac{1}{4\pi^2}\int_0^{\infty}\cos\left(\lambda\alpha+\frac{\kappa_0^2}{4\alpha}\right)d\alpha\\ &=\frac{1}{4\pi^2}\frac{\partial}{\partial\lambda}\int_0^{\infty}\sin\left(\lambda\alpha+\frac{\kappa_0^2}{4\alpha}\right)d\alpha. \qquad (A.17)\end{aligned}$$

In order to evaluate the last integral, we define a new variable of integration, according to

$$\alpha=\frac{\kappa_0}{2|\lambda|^{\frac{1}{2}}}e^{\vartheta}, \qquad (A.18)$$

whence

$$\begin{aligned}\int_0^{\infty}\sin\left(\lambda\alpha+\frac{\kappa_0^2}{4\alpha}\right)\frac{d\alpha}{\alpha}&=\int_{-\infty}^{\infty}\sin\left[\frac{\kappa_0|\lambda|^{\frac{1}{2}}}{2}\left(\frac{\lambda}{|\lambda|}e^{\vartheta}+e^{-\vartheta}\right)\right]d\vartheta\\ &=\begin{cases}\int_{-\infty}^{\infty}\sin(\kappa_0\lambda^{\frac{1}{2}}\cosh\vartheta)d\vartheta, & \lambda>0\\ -\int_{-\infty}^{\infty}\sin(\kappa_0(-\lambda)^{\frac{1}{2}}\sinh\vartheta)d\vartheta, & \lambda<0\end{cases} \qquad (A.19)\\ &=\begin{cases}\pi J_0(\kappa_0\lambda^{\frac{1}{2}}), & \lambda>0\\ 0, & 0<\lambda.\end{cases}\end{aligned}$$

This discontinuous value can be compactly represented by

$$\int_0^{\infty}\sin\left(\lambda\alpha+\frac{\kappa_0^2}{4\alpha}\right)\frac{d\alpha}{\alpha}=\pi ReH_0^{(1)}(\kappa_0\lambda^{\frac{1}{2}}) \qquad (A.20)$$

provided $\lambda^{\frac{1}{2}}$, with λ negative, is interpreted as $i|\lambda|^{\frac{1}{2}}$. Finally,

$$\bar{\Delta}(x)=\frac{1}{4\pi}\delta(\lambda)-\frac{\kappa_0^2}{8\pi}Re\frac{H_1^{(1)}(\kappa_0\lambda^{\frac{1}{2}})}{\kappa_0\lambda^{\frac{1}{2}}} \qquad (A.21)$$

where

$$Re\frac{H_1^{(1)}(\kappa_0\lambda^{\frac{1}{2}})}{\kappa_0\lambda^{\frac{1}{2}}}=\begin{cases}\frac{J_1(\kappa_0\lambda^{\frac{1}{2}})}{\kappa_0\lambda^{\frac{1}{2}}}, & \lambda>0\\ 0, & 0<\lambda,\end{cases} \qquad (A.22)$$

and the delta-function of λ arises from the discontinuity of (A.20) at $\lambda=0$. Clearly $\bar{\Delta}(x)$, and therefore $\Delta(x)$, vanishes if $x_\mu^2>0$ which is one of the defining properties of the latter function. On placing $\kappa_0=0$, we obtain

$$\begin{aligned}\bar{D}(x)=-\tfrac{1}{2}D(x)\epsilon(x)&=\frac{1}{4\pi}\delta(\lambda)\\ &=\frac{1}{4\pi}\delta(x_\mu^2), \qquad (A.23)\end{aligned}$$

which evidently corresponds to the propagation properties of electromagnetic pulses.

An integral representation for $\Delta(x)$ itself can be constructed with the aid of the Fourier inversion of (A.11):

$$\frac{a}{|a|}=-\frac{i}{\pi}P\int_{-\infty}^{\infty}e^{ia\tau}\frac{d\tau}{\tau}, \qquad (A.24)$$

in the form

$$\epsilon(x)=-\frac{\epsilon_\mu x_\mu}{|\epsilon_\mu x_\mu|}=\frac{i}{\pi}P\int_{\infty-}^{\infty}\exp(i\epsilon x_\mu\tau)\frac{d\tau}{\tau}. \qquad (A.25)$$

On employing the first expression of (A.12) for $\bar{\Delta}(x)$, we obtain

$$\begin{aligned}\Delta(x)=-\frac{2}{(2\pi)^5}\int (dk)\int_{-\infty}^{\infty}\frac{a}{|a|}daP\int_{-\infty}^{\infty}\frac{d\tau}{\tau}\\ \times\exp(i(k_\mu+\epsilon_\mu\tau)x_\mu)\exp(ia(k_\mu^2+\kappa_0^2)) \qquad (A.26)\end{aligned}$$

which becomes

$$\begin{aligned}\Delta(x)=-\frac{2}{(2\pi)^5}\int (dk)\int_{-\infty}^{\infty}\frac{a}{|a|}daP\int_{-\infty}^{\infty}\frac{d\tau}{\tau}\\ \times\exp(-2ia\epsilon_\mu k_\mu\tau)\exp(ia\epsilon_\mu^2\tau^2)\\ \times\exp(ik_\mu x_\mu)\exp(ia(k_\mu^2+\kappa_0^2)) \qquad (A.27)\end{aligned}$$

on introducing the transformation $k_\mu\rightarrow k_\mu-\epsilon_\mu\tau$. It may now be argued that (A.27) is independent of ϵ_μ, provided only that it is a time-like vector with positive ϵ_0. However, these essential characteristics can be maintained with $-\epsilon_\mu^2$ an arbitrarily small positive number. It is, therefore, permissible to evaluate (A.27) in the limit $\epsilon_\mu^2\rightarrow 0$, which, in view of the formula

$$\begin{aligned}P\int_{-\infty}^{\infty}\exp(-2ia\epsilon_\mu k_\mu\tau)\frac{d\tau}{\tau}&=-\pi i\frac{a}{|a|}\frac{\epsilon_\mu k_\mu}{|\epsilon_\mu k_\mu|}\\ &=\pi i\frac{a}{|a|}\epsilon(k), \qquad (A.28)\end{aligned}$$

yields

$$\begin{aligned}\Delta(x)&=-\frac{i}{(2\pi)^4}\int (dk)\int_{-\infty}^{\infty}da\exp(ia(k_\mu^2+\kappa_0^2))\\ &\qquad\times\exp(ik_\mu x_\mu)\epsilon(k)\\ &=-\frac{i}{(2\pi)^3}\int \exp(ik_\mu x_\mu)\delta(k_\mu^2+\kappa_0^2)\epsilon(k)(dk). \qquad (A.29)\end{aligned}$$

This result makes it evident that $\Delta(x)$, as constructed, satisfies the proper differential equation

$$(\square^2-\kappa_0^2)\Delta(x)=\frac{i}{(2\pi)^3}\int \exp(ik_\mu x_\mu)(k_\lambda^2+\kappa_0^2)\times\delta(k_\lambda^2+\kappa_0^2)\epsilon(k)(dk)=0, \tag{A.30}$$

since $x\delta(x)=0$. We have thereby completed the proof of the integral representation (A.10) since the three equations of definition for $\Delta(x)$ have been verified, the integral condition being equivalent, according to (A.5), to the differential equation defining $\bar{\Delta}(x)$.

An integral representation for $\Delta^{(1)}(x)$ can be obtained immediately from that of $\Delta(x)$. According to the equation of definition (1.57), and (A.29):

$$\begin{aligned}\Delta^{(1)}(x)&=\frac{1}{\pi}P\int_{-\infty}^{\infty}\Delta(x-\epsilon\tau)\frac{d\tau}{\tau}\\&=-\frac{i}{\pi}\frac{1}{(2\pi)^3}\int(dk)\epsilon(k)P\int_{-\infty}^{\infty}\frac{d\tau}{\tau}\times\exp(-ik_\mu\epsilon_\mu\tau)\exp(ik_\mu x_\mu)\delta(k_\mu^2+\kappa_0^2)\\&=\frac{1}{(2\pi)^3}\int\exp(ik_\mu x_\mu)\delta(k_\mu^2+\kappa_0^2)(dk),\end{aligned} \tag{A.31}$$

which is also easily derived, in a less formal manner, by means of the decomposition into positive and negative k_0. In order to evaluate $\Delta^{(1)}(x)$ in a manner similar to $\bar{\Delta}(x)$, we employ the integral representation

$$\delta(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ia\tau}da \tag{A.32}$$

with $\tau=k_\mu^2+\kappa_0^2$, and perform the integration over k space

$$\begin{aligned}\Delta^{(1)}(x)&=\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}da\int(dk)\exp(iak_\mu^2+ik_\mu x_\mu)\times\exp(ia\kappa_0^2)\\&=\frac{i}{16\pi^2}\int_{-\infty}^{\infty}\exp\left(-i\frac{x_\mu^2}{4a}+ia\kappa_0^2\right)\frac{a}{|a|}\frac{da}{a^2}\\&=\frac{i}{4\pi^2}\int_{-\infty}^{\infty}\exp\left(i\lambda\alpha+i\frac{\kappa_0^2}{4\alpha}\right)\frac{\alpha}{|\alpha|}d\alpha.\end{aligned} \tag{A.33}$$

Further rearrangements of (A.33) yield

$$\begin{aligned}\Delta^{(1)}(x)=\Delta^{(1)}(\lambda)&=-\frac{1}{2\pi^2}\int_0^{\infty}\sin\left(\lambda\alpha+\frac{\kappa_0^2}{4\alpha}\right)d\alpha\\&=\frac{1}{2\pi^2}\frac{\partial}{\partial\lambda}\int_0^{\infty}\cos\left(\lambda\alpha+\frac{\kappa_0^2}{4\alpha}\right)\frac{d\alpha}{\alpha}.\end{aligned} \tag{A.34}$$

Again utilizing the transformation (A.18), we now find

$$\begin{aligned}\int_0^{\infty}\cos\left(\lambda\alpha+\frac{\kappa_0^2}{4\alpha}\right)\frac{d\alpha}{\alpha}&=\begin{cases}\int_{-\infty}^{\infty}\cos(\kappa_0\lambda^{\frac{1}{2}}\cosh\vartheta)d\vartheta, & \lambda>0\\ \int_{-\infty}^{\infty}\cos(\kappa_0(-\lambda)^{\frac{1}{2}}\sinh\vartheta)d\vartheta, & \lambda<0\end{cases}\\&=\begin{cases}-\pi N_0(\kappa_0\lambda^{\frac{1}{2}}), & \lambda>0\\ 2K_0(\kappa_0(-\lambda)^{\frac{1}{2}}), & \lambda<0,\end{cases}\end{aligned} \tag{A.35}$$

which is summarized in

$$\int_0^{\infty}\cos\left(\lambda\alpha+\frac{\kappa_0^2}{4\alpha}\right)\frac{d\alpha}{\alpha}=-\pi Im H_0^{(1)}(\kappa_0\lambda^{\frac{1}{2}}). \tag{A.36}$$

Unlike the situation in (A.19), there is no discontinuity at $\lambda=0$. Therefore

$$\Delta^{(1)}(x)=\frac{\kappa_0^2}{4\pi}Im\frac{H_1^{(1)}(\kappa_0\lambda^{\frac{1}{2}})}{\kappa_0\lambda^{\frac{1}{2}}}=\begin{cases}\dfrac{\kappa_0^2}{4\pi}\dfrac{N_1(\kappa_0\lambda^{\frac{1}{2}})}{\kappa_0\lambda^{\frac{1}{2}}}, & \lambda>0\\ \dfrac{\kappa_0^2}{2\pi^2}\dfrac{K_1(\kappa_0(-\lambda)^{\frac{1}{2}})}{\kappa_0(-\lambda)^{\frac{1}{2}}}, & \lambda<0.\end{cases} \tag{A.37}$$

The singularity of $\Delta^{(1)}(x)$ at $\lambda=0$ can be exhibited by writing

$$\Delta^{(1)}(x)=-\frac{1}{2\pi^2\lambda}+\frac{\kappa_0^2}{4\pi}Im\left[\frac{H_1^{(1)}(\kappa_0\lambda^{\frac{1}{2}})}{\kappa_0\lambda^{\frac{1}{2}}}+\frac{2i}{\pi}\frac{1}{\kappa_0^2\lambda}\right] \tag{A.38}$$

since

$$Im\left[\frac{H_1^{(1)}(\kappa_0\lambda^{\frac{1}{2}})}{\kappa_0\lambda^{\frac{1}{2}}}+\frac{2i}{\pi}\frac{1}{\kappa_0^2\lambda}\right]\sim\frac{1}{\pi}\left[\log\frac{\gamma\kappa_0|\lambda|^{\frac{1}{2}}}{2}-\frac{1}{2}\right],\quad \kappa_0|\lambda|^{\frac{1}{2}}\ll 1 \tag{A.39}$$

where $\gamma=1.781$. On letting $\kappa_0\to 0$, we obtain

$$D^{(1)}(x)=-\frac{1}{2\pi^2\lambda}=\frac{1}{2\pi^2}\frac{1}{x_\mu^2}. \tag{A.40}$$