The renormalization group applications in QFT

- Perturbative RG: $SU(2)$ Yang–Mills
- An effective $SU(2)$ Yang–Mills theory at low energies
- Non-perturbative RG: Lamb shift in heavy atoms

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$SU(2)$ Yang–Mills theory with Lorenz gauge fixing

Semiclassical Lagrangian density

$$\begin{aligned}
\tilde{\mathcal{L}}_0^{tot} =& \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{1}{2\xi} (\partial_\mu A^a_\mu)^2 - \partial_\mu \bar{c}^a (D_\mu c)^a, \\
D_\mu c =& \partial_\mu c - ig[A_\mu, c], \\
F_{\mu\nu} =& \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu],
\end{aligned}$$

$\xi > 0$ is the Feynman parameter. $F_{\mu\nu}$, c, $\bar{c}$, A_μ are elements of the algebra, e.g. $A_\mu = t^a A^a_\mu$, $(t_c)^{ab} = -i\epsilon_{abc}$, $[t_a, t_b] = i\epsilon_{abc} t_c$.

The action $\int \tilde{\mathcal{L}}_0^{tot}$ is invariant under the transformation

$$\delta^{BRS} A_\mu = \epsilon\, D_\mu c, \qquad \delta^{BRS} c = \epsilon\, \frac{1}{2} ig\{c, c\}, \qquad \delta^{BRS} \bar{c} = -\epsilon\, \frac{1}{\xi} \partial A\,,$$

where ϵ is a Grassmann parameter, and $\{c, c\}^d = i\epsilon_{abd} c^a c^b$.

Regulators

Λ, Λ_0 are IR and UV cutoffs, $0 < \Lambda \leqslant \Lambda_0$.

$$C^{\Lambda\Lambda_0}_{\mu\nu} = \frac{1}{p^2}(\delta_{\mu\nu} + (\xi - 1)\frac{p_\mu p_\nu}{p^2})\,\sigma_{\Lambda\Lambda_0}(p^2)\,,$$

$$S^{\Lambda\Lambda_0}(p) = \frac{1}{p^2}\,\sigma_{\Lambda\Lambda_0}(p^2), \quad \sigma_{\Lambda\Lambda_0}(p^2) = e^{-\frac{p^4}{\Lambda_0^4}} - e^{-\frac{p^4}{\Lambda^4}}\,.$$

Recover the usual covariances in the limit $\Lambda \to 0$, $\Lambda_0 \to \infty$.

Def: a Gaussian measure $d\tilde{\mu}(A, c, \bar{c})$ with the characteristic function:

$$e^{\frac{1}{\hbar}\langle\bar{\eta}, S^{\Lambda\Lambda_0}\eta\rangle - \frac{1}{2\hbar}\langle j, C^{\Lambda\Lambda_0} j\rangle} = \int d\tilde{\mu}_{\Lambda\Lambda_0}(\Phi) e^{\frac{i}{\hbar}\langle\Phi, K\rangle}\,,$$

where $\Phi = (A, c, \bar{c})$ and $K = (j, \bar{\eta}, \eta)$.

The counterterms

All counterterms respecting the global symmetries and having ghost number zero

$$
\begin{aligned}
\mathcal{L}_{ct}^{\Lambda_0\Lambda_0} &= r^{\bar{c}c\bar{c}c}\bar{c}^b c^b \bar{c}^a c^a + r_1^{\bar{c}cAA}\bar{c}^b c^b A_\mu^a A_\mu^a + r_2^{\bar{c}cAA}\bar{c}^a c^b A_\mu^a A_\mu^b \\
&+ r_1^{A^4} A_\mu^b A_\nu^b A_\mu^a A_\nu^a + r_2^{A^4} A_\nu^b A_\nu^b A_\mu^a A_\mu^a + 2\epsilon_{abc} r^{A^3}(\partial_\mu A_\nu^a) A_\mu^b A_\nu^c \\
&- r_1^{A\bar{c}c}\epsilon_{abd}(\partial_\mu \bar{c}^a) A_\mu^b c^d - r_2^{A\bar{c}c}\epsilon_{abd}\bar{c}^a A_\mu^b \partial_\mu c^d + \Sigma^{\bar{c}c}\bar{c}^a \partial^2 c^a \\
&- \frac{1}{2}\Sigma_T^{AA} A_\mu^a(\partial^2\delta_{\mu\nu} - \partial_\mu\partial_\nu) A_\nu^a + \frac{1}{2\xi}\Sigma_L^{AA}(\partial_\mu A_\mu^a)^2 \\
&+ \delta m_{AA}^2 A_\mu^a A_\mu^a + \delta m_{\bar{c}c}^2 \bar{c}^a c^a .
\end{aligned}
$$

There are eleven marginal counterterms which depend on Λ_0, $\delta m_{\bar{c}c}^2 = 0$. Expansion in powers $\hbar$:

$$\mathcal{L}_{ct} = \sum_{l>0} \hbar^l \mathcal{L}_{ct;l}$$

The BRST symmetry

To obtain the antighost equation (AGE) we need an auxiliary field. Full tree level Lagrangian density in the limit $\Lambda \to 0$, $\Lambda_0 \to \infty$ is

$$\mathcal{L}_0^{tot} = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{\xi}{2} B^2 - iB\partial_\mu A_\mu - \partial_\mu \bar{c} D_\mu c \; .$$

The action $\int \mathcal{L}_0^{tot}$ is invariant under the infinitesimal transformation

$$\delta^{BRS} A = \epsilon\, Dc, \qquad \delta^{BRS} c = \epsilon\, \frac{1}{2} ig \{c, c\},$$

$$\delta^{BRS} \bar{c} = \epsilon\, iB, \qquad \delta^{BRS} B = 0,$$

Defining the classical BRST (Becchi, Rouet, Stora, Tyutin) operator s

$$\delta^{BRS} \Phi = \epsilon\, s\, \Phi, \quad \Longrightarrow \quad s^2 = 0.$$

BRST invariance is explicitly broken by the regulators.

Generating functionals

$$d\mu_{\Lambda\Lambda_0}(A, B, \dots) = d\nu_{\Lambda\Lambda_0}(A) d\upsilon_{\Lambda\Lambda_0}(B - i\frac{1}{\xi}\partial A) \dots$$

$$Z^{\Lambda\Lambda_0}(K) = \int d\mu_{\Lambda\Lambda_0}(\Phi)\, e^{-\frac{1}{\hbar} L^{\Lambda_0\Lambda_0}}\, e^{\frac{1}{\hbar}\langle K, \Phi\rangle}$$

$$\mathcal{L}^{\Lambda_0\Lambda_0} = g\epsilon_{abc}\partial_\mu A_\nu^a A_\mu^b A_\nu^c + \frac{g^2}{4}\epsilon_{cab}\epsilon_{cds} A_\mu^a A_\nu^b A_\mu^d A_\nu^s - g\epsilon_{abc}\partial_\mu \bar{c}^a A_\mu^b c^c + \mathcal{L}_{ct}^{\Lambda_0\Lambda_0}$$

The tree level interaction does not depend on the B field.
Generator of Connected Schwinger functions:

$$W^{\Lambda\Lambda_0} = \hbar \log Z^{\Lambda\Lambda_0}$$

Generator of Connected Amputated Schwinger functions:

$$L^{\Lambda\Lambda_0}(\Phi) = -\hbar \log \int d\mu_{\Lambda\Lambda_0}(\Phi')\, e^{-\frac{1}{\hbar} L^{\Lambda_0\Lambda_0}(\Phi' + \Phi)}$$

The effective action

The effective action is the Legendre transform of W

$$\Gamma^{\Lambda\Lambda_0}(\Phi) = \langle K, \Phi \rangle - W^{\Lambda\Lambda_0}(K)\,.$$

Reduced effective action:

$$\Gamma^{\Lambda\Lambda_0}(\Phi) = \Gamma^{\Lambda\Lambda_0}(\Phi) - \frac{1}{2}\langle \Phi, \mathbf{C}^{-1}_{\Lambda\Lambda_0}\Phi \rangle$$

Expansion in powers $\hbar$:

$$\Gamma = \sum_{l=0}^{\infty} \hbar^l \Gamma_l, \qquad \Gamma_{l=0}^{\Lambda\Lambda_0;\phi\phi} = 0$$

Here $\delta(\sum_{i=0}^{n-1} p_i)\Gamma^{\vec{\phi};w}(\vec{p}) := \partial^w \Big(\prod_{i=0}^{n-1} \frac{\delta}{\delta\phi_i(p_i)} \Big)\Gamma \Big|_{\vec{\phi}=0}$, $w = (0, w_1, ..., w_{n-1})$.

Main idea

The characteristic function is the Fourier transform of the measure:

$$\chi(j) = \mathcal{N} \int [d\phi] \, e^{-\frac{1}{2\hbar}\langle \phi C^{-1}_{\Lambda\Lambda_0} \phi\rangle + i\frac{1}{\hbar}\langle j\phi\rangle}$$
$$= \int d\mu(\phi) e^{i\frac{1}{\hbar}\langle j\phi\rangle} = e^{-\frac{1}{2\hbar}\langle j C_{\Lambda\Lambda_0} j\rangle}$$

It is used to obtain all moments:

$$E\big(\phi(x_0), ..., \phi(x_{N-1})\big) = \prod_{i=0}^{N-1} \frac{\hbar}{i} \frac{\delta}{\delta j(x_i)} \chi(j)\Big|_{j=0}$$

Important differential equation:

$$\frac{d}{d\Lambda}\chi(j) = \dot{\chi}(j) = -\frac{1}{2\hbar}\langle j \dot{C}_{\Lambda\Lambda_0} j\rangle e^{\frac{1}{2\hbar}\langle j C_{\Lambda\Lambda_0} j\rangle}$$
$$= \frac{\hbar}{2} \int d\mu(\phi) \langle \frac{\delta}{\delta\phi} \dot{C}_{\Lambda\Lambda_0} \frac{\delta}{\delta\phi}\rangle e^{i\frac{1}{\hbar}\langle j\phi\rangle}$$

RG equation

Wilson; Wegner; Polchinski...

$$\dot{L} = \frac{\hbar}{2}\langle\frac{\delta}{\delta\Phi}, \dot{\mathbf{C}}\frac{\delta}{\delta\Phi}\rangle L - \frac{1}{2}\langle\frac{\delta L}{\delta\Phi}, \dot{\mathbf{C}}\frac{\delta L}{\delta\Phi}\rangle \qquad \dot{L} := \partial_\Lambda L^{\Lambda\Lambda_0}$$

Wetterich; Morris; Bonini, D'Attanasio, Marchesini...

$$\dot{\Gamma} = \frac{\hbar}{2}\langle\dot{\mathbf{C}}, \mathrm{L}''\rangle \qquad \mathrm{L}'' = \Gamma''(1 + \hat{\mathbf{1}}\mathbf{C}\Gamma'')^{-1}$$

$$\dot{\Gamma} = \frac{\hbar}{2}\langle\dot{\mathbf{C}}\,\Gamma''\sum_{m=0}^{\infty}\left(-\hat{\mathbf{1}}\mathbf{C}\,\Gamma''\right)^m\rangle \qquad \hat{\mathbf{1}} = \begin{cases}\delta_{\varphi^a\varphi^b} & \\ -\delta_{c_a c_b} & -\delta_{\bar{c}_a\bar{c}_b}\end{cases}$$

Trivial dependence on the B field (linear gauge)

Vanishing renormalization conditions for all relevant terms,

$$\partial^w \Gamma_l^{0\Lambda_0;B\vec{\phi}}(\vec{q}) = 0\,,$$

imply that $\forall\Lambda > 0$

$$\Gamma_l^{\Lambda\Lambda_0;B\vec{\phi}}(\vec{p}) = 0\,.$$

Thus there are no counterterms with the field B.
Dependence on the B-fields known explicitly:

$$\Gamma^{\Lambda\Lambda_0}(A,B,c,\bar{c}) = \frac{1}{2\xi}\int d^4x\,(\xi B - i\partial A)^2 + \tilde{\Gamma}^{\Lambda\Lambda_0}(A,c,\bar{c})\,.$$

where $\tilde{\Gamma}^{\Lambda\Lambda_0}$ is defined using the measure $d\tilde{\mu}(A,c,\bar{c})$.

Violated Slavnov–Taylor Identities (VSTI)

Introduce the Lagrangian density

$$\mathcal{L}_{vst}^{\Lambda_0\Lambda_0} = \mathcal{L}^{\Lambda_0\Lambda_0} + \gamma\psi^{\Lambda_0} + \omega\Omega^{\Lambda_0},$$

where γ, ω are external sources, $R_i^{\Lambda_0} = 1 + O(\hbar)$,

$$\psi^{\Lambda_0} = R_1^{\Lambda_0}\partial c - igR_2^{\Lambda_0}[A,c], \qquad \Omega^{\Lambda_0} = \frac{1}{2i}gR_3^{\Lambda_0}\{c,c\}\,.$$

We consider the change of variables $\Phi \mapsto \Phi + \delta_\epsilon\Phi$

$$\delta_\epsilon A = \epsilon\,\sigma_{0\Lambda_0}\psi^{\Lambda_0}, \qquad \delta_\epsilon c = -\epsilon\,\sigma_{0\Lambda_0}\Omega^{\Lambda_0}, \qquad \delta_\epsilon\bar{c} = \epsilon\,\sigma_{0\Lambda_0} iB\,,$$

which leaves invariant the functional integral: $\delta_\epsilon Z^{0\Lambda_0} = 0$, with

$$Z^{0\Lambda_0}(K) = \int d\mu_{0\Lambda_0}(\Phi)\, e^{-\frac{1}{\hbar}L_{vst}^{\Lambda_0\Lambda_0}}\, e^{\frac{1}{\hbar}\langle K,\Phi\rangle}.$$

Performing the Legendre transform the identity $\delta_\epsilon Z^{0\Lambda_0} = 0$ gives

$$\Gamma_1^{0\Lambda_0} + \int d^4x\, (iB + \frac{1}{\xi}\partial A)\Gamma_\beta^{0\Lambda_0} = \frac{1}{2}\mathcal{S}\underline{\Gamma}^{0\Lambda_0},$$

where we have introduced an auxiliary functional $\underline{\Gamma}^{0\Lambda_0}$

$$\underline{\Gamma}^{0\Lambda_0} = i\langle B, \bar{\omega}\rangle + \underline{\tilde{\Gamma}}^{0\Lambda_0}, \qquad \underline{\tilde{\Gamma}}^{0\Lambda_0} = \tilde{\Gamma}^{0\Lambda_0} + \frac{1}{2\xi}\langle A, \partial\partial\, A\rangle\,,$$

and the quantum BRST operator $\mathcal{S} = \mathcal{S}_{\tilde{c}} + \mathcal{S}_A - \mathcal{S}_c$, with

$$\mathcal{S}_\phi = \langle\frac{\delta\underline{\Gamma}^{0\Lambda_0}}{\delta\phi}, \sigma_{0\Lambda_0}\frac{\delta}{\delta\phi^\ddagger}\rangle + \langle\frac{\delta\underline{\Gamma}^{0\Lambda_0}}{\delta\phi^\ddagger}, \sigma_{0\Lambda_0}\frac{\delta}{\delta\phi}\rangle,$$

$$(\phi, \phi^\ddagger) \in \{(A,\gamma), (c,\omega), (\tilde{c},\bar{\omega})\}, \qquad \frac{\delta}{\delta\tilde{c}} = \frac{\delta}{\delta\bar{c}} - \partial\frac{\delta}{\delta\gamma}\,.$$

Nilpotency of the BRST

We rewrite the VSTI in the following form

$$\langle iB, \tilde{\Gamma}^{0\Lambda_0}_{\beta} \rangle = \frac{1}{2} \mathcal{S}_{\tilde{c}} \underline{\Gamma}^{0\Lambda_0} = \langle iB, \sigma_{0\Lambda_0} \frac{\delta}{\delta \tilde{c}} \tilde{\underline{\Gamma}}^{0\Lambda_0} \rangle,$$
$$\tilde{\mathrm{F}}^{0\Lambda_0}_{1} = \frac{1}{2} \tilde{S} \tilde{\underline{\Gamma}}^{0\Lambda_0},$$

where

$$\tilde{S} = \mathcal{S}_A - \mathcal{S}_c, \qquad \tilde{\mathrm{F}}^{0\Lambda_0}_{1} = \tilde{\Gamma}^{0\Lambda_0}_{1} + \frac{1}{\xi} \langle \partial A, \tilde{\Gamma}^{0\Lambda_0}_{\beta} \rangle \,.$$

Important properties of $\mathcal{S}_\phi$: $\forall \phi, \phi' \in \{A, c, \tilde{c}\}$

$$(\mathcal{S}_\phi \mathcal{S}_{\phi'} + \mathcal{S}_{\phi'} \mathcal{S}_\phi) \underline{\Gamma}^{0\Lambda_0} = 0.$$

It follows that $\tilde{S}^2 \tilde{\underline{\Gamma}}^{0\Lambda_0} = 0$, $\tilde{S} \mathcal{S}_{\tilde{c}} = -\mathcal{S}_{\tilde{c}} \tilde{S}$. The nilpotency implies:

$$\tilde{S} \tilde{\mathrm{F}}_1 = 0, \qquad \tilde{S} \tilde{\Gamma}_{\beta} + \sigma_{0\Lambda_0} \left(\frac{\delta}{\delta \bar{c}} - \partial \frac{\delta}{\delta \gamma} \right) \tilde{\mathrm{F}}_1 = 0.$$

Explicit form of the VSTI

$$\tilde{\Gamma}^{0\Lambda_0}_{\beta} = \sigma_{0\Lambda_0}\left(\frac{\delta\underline{\tilde{\Gamma}}^{0\Lambda_0}}{\delta\bar{c}} - \partial\underline{\tilde{\Gamma}}^{0\Lambda_0}_{\gamma}\right) \qquad (AGE)$$

$$\tilde{\mathrm{F}}^{0\Lambda_0}_{1} = \langle\frac{\delta\underline{\tilde{\Gamma}}^{0\Lambda_0}}{\delta A}, \sigma_{0\Lambda_0}\underline{\tilde{\Gamma}}^{0\Lambda_0}_{\gamma}\rangle - \langle\frac{\delta\underline{\tilde{\Gamma}}^{0\Lambda_0}}{\delta c}, \sigma_{0\Lambda_0}\underline{\tilde{\Gamma}}^{0\Lambda_0}_{\omega}\rangle \qquad (STI)$$

The goal is to show that $\tilde{\Gamma}^{0\infty}_{\beta} = 0$, $\tilde{\mathrm{F}}^{0\infty}_{1} = 0$ which imply $\tilde{\Gamma}^{0\infty}_{1} = 0$.

Another form of the VSTI

$$\tilde{\Gamma}^{0\Lambda_0}_{1} = \langle\frac{\delta\tilde{\Gamma}^{0\Lambda_0}}{\delta A}, \sigma_{0\Lambda_0}\tilde{\Gamma}^{0\Lambda_0}_{\gamma}\rangle - \langle\frac{\delta\tilde{\Gamma}^{0\Lambda_0}}{\delta c}, \sigma_{0\Lambda_0}\tilde{\Gamma}^{0\Lambda_0}_{\omega}\rangle - \frac{1}{\xi}\langle\partial A, \sigma_{0\Lambda_0}\frac{\delta\tilde{\Gamma}^{0\Lambda_0}}{\delta\bar{c}}\rangle\,.$$

The vertex functions on the rhs give renormalization conditions for the anomalies on the lhs.

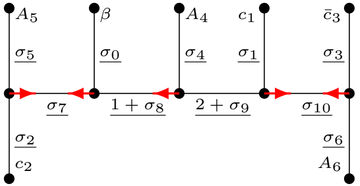
Renormalization conditions

No tilde on $\Gamma(A, c, \bar{c})$

- $\partial^w \Gamma^{0\Lambda_0;\vec{\phi}}_{\vec{\varkappa}}(0) = 0$, for all strictly relevant terms, $\varkappa \in \{\gamma, \omega\}$.
- $\Gamma^{M\Lambda_0;c\bar{c}c\bar{c}}(0) = 0$, $\Gamma^{M\Lambda_0;c\bar{c}A^2}(0) = 0$, $\partial_A \Gamma^{M\Lambda_0;c\bar{c}A}(0) = 0$. We show that the counterterms $r^{\bar{c}c\bar{c}c}$, $r_1^{\bar{c}cA^2}$, $r_2^{\bar{c}cA^2}$, $r_2^{A\bar{c}c}$ vanish.
- The renormalization constants r^{A^3} , Σ_T^{AA} , $\Sigma^{\bar{c}c}$ are free.
- The remaining 7 renormalization constants must satisfy 7 additional relations in order to make the marginal terms $\Gamma_1^{\vec{\phi};w}$, $\Gamma_\beta^{\vec{\phi};w}$ at the renormalization point comply with the bounds **3** and **4** below.
- We prove the existence of a solution for this system of relations that does not depend on the UV cutoff.

Trees $\mathcal{T}_{\vec{\varphi}}$ and $\mathcal{T}_{1\vec{\varphi}}$

- Vertices of valence one and three, $V = V_1 \cup V_3$.
- An edge e carries momentum p_e. Momentum conservation at the vertices.
- To any edge are associated $\rho(e) \in \{0, 1, 2\}$ and the number of momentum derivatives $\sigma(e)$.
- An edge has a θ-weight, $\theta(e) = \rho(e) + \sigma(e)$.



$$\chi : V_\bullet \to E \backslash E_1$$

$$\rho(e) = 2 - |\chi^{-1}(e)|$$

$$\forall \tau \in \mathcal{T}_{\phi_0,\dots\phi_{n-1}},\ n \geqslant 4:\ \theta(\tau) = n + \|w\| - 4,\ \theta(\tau) = -[\Gamma^{\vec{\phi};w}]$$

Main elements of the bounds

For a tree τ we sum over the family of θ-weights $\Theta_\tau^w = \{\theta_j(e)\}_j$.

$$\Pi^{\Lambda}_{\tau,\theta}(\vec{p}) = \prod_{e\in E} \frac{1}{(\Lambda + |p_e|)^{\theta(e)}}\,,$$

$$Q^{\Lambda;w}_{\tau}(\vec{p}) = \begin{cases} \displaystyle\sum_{\Theta_\tau^w} \Pi^{\Lambda}_{\tau,\theta}(\vec{p}), & |V_1| > 3\,, \\ \displaystyle\inf_{i\in\mathbb{I}} \sum_{\Theta_\tau^{w'(i)}} \Pi^{\Lambda}_{\tau,\theta}(\vec{p}), & |V_1| = 3\,. \end{cases}$$

$w'(i)$ is obtained from w by diminishing w_i by one unit.
For nonvanishing w $\mathbb{I} = \{i : w_i > 0\}$.

$$\eta(\vec{p}) = \min_{\mathbb{I}\in\wp_{n-1}\setminus\{\emptyset\}} (|\sum_{i\in\mathbb{I}} p_i|, M)$$

Bounds on vertex functions

Let $d = 4 - 2n_{\varkappa} - \mathrm{N} - \|w\|$.

1.a) $d \geqslant 0$ or $\mathrm{N} + n_{\varkappa} = 2$

$$|\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{\vec{\varkappa};l}(\vec{p})| \leqslant (\Lambda + |\vec{p}|)^d P^{\Lambda\Lambda}_r(\vec{p}) \,,$$

1.b) $d < 0$

$$|\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{\vec{\varkappa};l}(\vec{p})| \leqslant \sum_{\tau \in \mathcal{T}_{\vec{\varkappa}\vec{\phi}}} Q^{\Lambda;w}_{\tau}(\vec{p}) \; P^{\Lambda\Lambda}_r(\vec{p}) \,,$$

$$P^{\Lambda\Lambda'}_r(\vec{p}) = \mathcal{P}^{(0)}_r\left(\log_+ \frac{\max(|\vec{p}|, M)}{\Lambda + \eta(\vec{p})}\right) + \mathcal{P}^{(1)}_r\left(\log_+ \frac{\Lambda'}{M}\right).$$

$\mathcal{P}_r$ denotes polynomials with nonnegative coefficients and degree

$$r = \begin{cases} 2l & d \geqslant 0 \\ 2l - 1 & d < 0 \end{cases}$$

UV convergence

Let $d = 4 - 2n_{\varkappa} - \mathrm{N} - \|w\|$.

2.a) $d \geqslant 0$ or $\mathrm{N} + n_{\varkappa} = 2$

$$|\partial_{\Lambda_0} \Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{\vec{\varkappa};l}(\vec{p})| \leqslant \frac{\Lambda + M + |\vec{p}|}{\Lambda_0^2} (\Lambda + |p|)^d P_r^{\Lambda\Lambda_0}(\vec{p}) \,,$$

2.b) $d < 0$

$$|\partial_{\Lambda_0} \Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{\vec{\varkappa};l}(\vec{p})| \leqslant \frac{\Lambda + M + |\vec{p}|}{\Lambda_0^2} \sum_{\tau \in \mathcal{T}_{\vec{\varkappa}\vec{\phi}}} Q_\tau^{\Lambda;w}(\vec{p})\, P_r^{\Lambda\Lambda_0}(\vec{p}) \,.$$

Cauchy criterion

$$|\Gamma^{\Lambda\Lambda_0';\vec{\phi};w}_{\vec{\varkappa};l} - \Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{\vec{\varkappa};l}| \leqslant \int_{\Lambda_0}^{\Lambda_0'} d\lambda_0 \, |\partial_{\lambda_0} \Gamma^{\Lambda\lambda_0;\vec{\phi};w}_{\vec{\varkappa};l}| \,.$$

Restoration of AGE

Let $d = 3 - 2n_\varkappa - \mathrm{N} - \|w\|$.

3.a) $d \geqslant 0$ or $\mathrm{N} + n_\varkappa = 1$

$$|\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{\beta\vec{\varkappa};l}(\vec{p})| \leqslant \frac{M + |\vec{p}| + \Lambda}{\Lambda_0}(\Lambda + |\vec{p}|)^d P^{\Lambda\Lambda_0}_{r\,0}(\vec{p})\,,$$

3.b) $d < 0$

$$|\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{\beta\vec{\varkappa};l}(\vec{p})| \leqslant \frac{M + |\vec{p}| + \Lambda}{\Lambda_0} \sum_{\tau \in \mathcal{T}_{\beta\vec{\varkappa}\vec{\phi}}} Q^{\Lambda;w}_\tau(\vec{p})\, P^{\Lambda\Lambda_0}_{r\,0}(\vec{p})\,,$$

$$P^{\Lambda\Lambda_0}_{r\,s}(\vec{p}) = \left(1 + \left(\frac{|\vec{p}|}{\Lambda_0}\right)^4\right)\mathcal{P}^{(2)}_s\left(\frac{|\vec{p}|}{\Lambda + M}\right) P^{\Lambda\Lambda_0}_r(\vec{p})\,.$$

Restoration of STI

Let $d = 5 - 2n_{\varkappa} - \mathrm{N} - \|w\|$.

4.a) $d > 0$ or $\mathrm{N} + n_{\varkappa} = 2$

$$|\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{1\vec{\varkappa};l}(\vec{p})| \leqslant \frac{M + |\vec{p}| + \Lambda}{\Lambda_0}(\Lambda + |\vec{p}|)^d P^{\Lambda\Lambda_0}_{r_1\, s_1}(\vec{p})\,,$$

4.b) $d \leqslant 0$

$$|\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{1\vec{\varkappa};l}(\vec{p})| \leqslant \frac{M + |\vec{p}| + \Lambda}{\Lambda_0} \sum_{\tau \in \mathcal{T}_{1\vec{\varkappa}\vec{\phi}}} Q^{\Lambda;w}_{\tau}(\vec{p})\, P^{\Lambda\Lambda_0}_{r_1\, s_1}(\vec{p})\,.$$

r_1, s_1 are linear functions of loop number l.

Overview of the proof

We proceed by induction in the loop order l: $\Gamma = \sum_{l=0}^{\infty} \hbar^l \Gamma_l$.

The irrelevant terms are constructed by integrating the FE from Λ_0 down to Λ:

$$\Gamma_l^{\Lambda\Lambda_0;\vec{\phi};w} = \Gamma_l^{\Lambda_0\Lambda_0;\vec{\phi};w} + \int_{\Lambda_0}^{\Lambda} d\lambda\, \dot{\Gamma}_l^{\lambda\Lambda_0;\vec{\phi};w}, \quad \Gamma_l^{\Lambda_0\Lambda_0;\vec{\phi};w} = 0$$

$$\dot{\Gamma}^{\vec{\phi}} = \frac{\hbar}{2} \sum_{\vec{\phi}_1,\ldots,\vec{\phi}_m} \langle \dot{\mathbf{C}}_{\zeta_1\bar{\zeta}_m} \Gamma^{\zeta_1\vec{\phi}_1\bar{\zeta}_1} \prod_{j=2}^{m} \mathbf{C}_{\zeta_j\bar{\zeta}_{j-1}} \Gamma^{\zeta_j\vec{\phi}_j\bar{\zeta}_j} \rangle$$

Here $\Gamma_{l=0}^{\zeta\bar{\zeta}} = 0$.

For the relevant terms we integrate the FE from $\Lambda = 0$ up to arbitrary Λ at the renormalization point $\vec{q}$

$$\Gamma_l^{\Lambda\Lambda_0;\vec{\phi};w}(\vec{q}) = \Gamma_l^{0\Lambda_0;\vec{\phi};w}(\vec{q}) + \int_0^{\Lambda} d\lambda \, \dot{\Gamma}_l^{\lambda\Lambda_0;\vec{\phi};w}(\vec{q})$$

We interpolate this to arbitrary momenta $\vec{p}$ integrating over the corresponding irrelevant terms;

$$\Gamma_l^{\Lambda\Lambda_0;\vec{\phi};w}(\vec{p}) = \Gamma_l^{\Lambda\Lambda_0;\vec{\phi};w}(\vec{q}) + \int_{\vec{q}}^{\vec{p}} \partial\Gamma_l^{\Lambda\Lambda_0;\vec{\phi};w}$$

Renormalization conditions on Γ_1

$$\Gamma_1^{cAAA} \sim \Gamma^{AAAA}\Gamma_\gamma^{c} + \sum_{Z_3} \Gamma^{AAA}\Gamma_\gamma^{Ac} + \underline{\Gamma}^{AA}\Gamma_\gamma^{cAA}$$

$\sim$ stands for equality up to terms $\frac{M}{\Lambda_0}\log^{r_1}\frac{\Lambda_0}{M}$.

5 equations u_i^{cAAA} and 2 variables R_1^{AAAA}, R_2^{AAAA}

$$u_{1,2}^{cAAA} = 8R_1 R_2^{AAAA} - 4gR_2 R^{AAA} + \zeta_{1,2}^{\Lambda_0} \sim 0$$
$$u_{3,4,5}^{cAAA} = 4R_1 R_1^{AAAA} + 2gR_2 R^{AAA} + \zeta_{3,4,5}^{\Lambda_0} \sim 0$$

Using AGE $\Gamma_\beta \sim 0$

$$\begin{matrix} \tilde{S}\Gamma_1 \sim 0 \\ \Gamma_1^{\bar{c}} - \partial\Gamma_{1;\gamma} \sim 0 \end{matrix} \implies \begin{matrix} u_1^{cAAA} \sim u_2^{cAAA} \\ u_3^{cAAA} \sim u_4^{cAAA} \sim u_5^{cAAA} \end{matrix}$$

Maximal Abelian Gauge (MAG)

Extremum of $(A^1)^2 + (A^2)^2$ over the gauge transformations.

$$A^{\pm} = \frac{A^2 \pm i A^1}{\sqrt{2}}, \qquad A = A^3$$

Gauge transformation has the following form:

$$\delta A = \partial\alpha - ig(A^+\alpha^- - A^-\alpha^+), \qquad D^{\pm}_{\mu} A^{\pm}_{\nu} = \partial_\mu A^{\pm}_{\nu} \mp ig A_\mu A^{\pm}_{\nu}$$

$$\delta A^{\pm} = D^{\pm}\alpha^{\pm} \pm ig\alpha A^{\pm}$$

extremum A^+A^- under $\alpha^{\pm} \implies D^{\pm}_{\nu} A^{\pm}_{\nu} = 0$
We fix $U(1)$ symmetry by $\partial A = 0$. The t'Hooft gauge fixing Lagrangian density is

$$\mathcal{L}^{GF} = \frac{1}{2\xi}(\partial A)^2 + \frac{1}{\xi}(D^+_{\nu} A^+_{\nu})(D^-_{\mu} A^-_{\mu})$$

With the B^a field we have $\mathcal{L}^{GF} = \mathcal{L}_2^{GF} + \mathcal{L}_3^{GF}$ where

$$\mathcal{L}_2^{GF} = \frac{\xi}{2} B^2 + \xi B^+ B^- - i B^a \partial_\nu A_\nu^a, \qquad a \in (+,-,3)$$
$$\mathcal{L}_3^{GF} = g B^+ A_\nu A_\nu^- - g B^- A_\nu A_\nu^+, \qquad B = B^3$$

$L^{YM} + L^{FP} + L^{GF}$ is BRST invariant, i.e. $\delta\Phi = \theta\, s\Phi$

$$sA = \partial c - ig(A^+ c_+ - A^- c_+), \qquad sc = igc_+ c_-, \qquad s\bar{c}_a = iB^a$$
$$sA^\pm = D^\pm c_\pm \pm igcA^\pm, \qquad sc_\pm = \pm igcc_\pm, \qquad sB^a = 0$$

Regularization: $\sigma_\lambda(p^2) = e^{-\left(\frac{p^2}{\lambda^2}\right)^n}$; BRST insertions:

$$\psi = R_1^{\Lambda_0} \partial c + igR_2^{\Lambda_0}(A^+ c_- - A^- c_+), \qquad \Omega = igR_3 c_- c_+$$
$$\psi^\pm = R_4^{\Lambda_0} \partial c_\pm \mp igR_5^{\Lambda_0} A c_\pm \pm igR_6^{\Lambda_0} c A^\pm, \qquad \Omega^\pm = \pm igR_7 c_\pm c$$

VSTI

$$\lim_{\Lambda_0\to\infty}\Big(\sigma_{0\Lambda_0}\frac{\delta \mathrm{F}}{\delta \bar{c}} - \partial\Gamma_\gamma\Big) = 0 \qquad (AGE)$$

$$\lim_{\Lambda_0\to\infty}\Big(\langle\frac{\delta \mathrm{F}}{\delta A^a}\sigma_{0\Lambda_0}\Gamma_{\gamma^{\check{a}}}\rangle - \langle\frac{\delta \mathrm{F}}{\delta c_a}\sigma_{0\Lambda_0}\Gamma_{\omega^{\check{a}}}\rangle - \frac{1}{\xi}\langle\partial A\sigma_{0\Lambda_0}\frac{\delta \mathrm{F}}{\delta \bar{c}}\rangle + i\langle B^{\pm}\sigma_{0\Lambda_0}\frac{\delta \mathrm{F}}{\delta \bar{c}_{\pm}}\rangle + \hbar\Delta^{\Lambda\Lambda_0}\Big) = 0 \qquad (STI)$$

$$\mathrm{F}^{\Lambda\Lambda_0} = \frac{1}{2}\langle\Phi\mathbf{C}^{-1}_{0\Lambda_0}\Phi\rangle + \Gamma^{\Lambda\Lambda_0},$$

$$\Delta^{\Lambda\Lambda_0} = -\langle\sigma_\Lambda(1+\Gamma''\hat{\mathbf{C}})^{-1}_{A^a\Phi}\frac{\delta\Gamma_{\gamma^{\check{a}}}}{\delta\Phi}\rangle - \langle\sigma_\Lambda\partial(1+\Gamma''\hat{\mathbf{C}})^{-1}_{A^a c_a}\rangle + \langle\sigma_\Lambda(1+\Gamma''\hat{\mathbf{C}})^{-1}_{c_a\Phi}\frac{\delta\Gamma_{\omega^{\check{a}}}}{\delta\Phi}\rangle,$$

$$\Gamma'' = \frac{\delta^2\Gamma}{\delta\Phi\delta\Phi'}, \quad \hat{\mathbf{C}} = \hat{\mathbf{1}}\mathbf{C}, \quad \hat{\mathbf{1}} = \begin{cases}\delta_{\varphi^a\varphi^b} & \\ -\delta_{c_a c_b} & -\delta_{\bar{c}_a\bar{c}_b}\end{cases}$$

Mass term of the condensate

$$\Gamma^{\Lambda\Lambda_0} = \langle A^+_\mu (p^2 \delta_{\mu\nu} - p_\mu p_\nu) \Sigma^{\Lambda\Lambda_0} A^-_\nu \rangle + \langle A^+ \delta m^2_{\Lambda\Lambda_0} A^- \rangle + \dots$$

Consider the vertex function $\Gamma_1^{A^+ c_-}$. Using VSTI we have

$$\mathrm{F}^{A^+_\beta A^-} \partial_{p_\alpha} \Gamma^{c_-}_{\gamma^+} + \partial_{p_\alpha} \Delta^{A^+_\beta c_-}(p) \Big|_{p=0} = \delta_{\beta\alpha} \delta m^2 R_4 + i \frac{\partial \Delta^{A^+_\beta c_-}(p)}{\partial p_\alpha} \Big|_{p=0} = 0$$

Substituting the tree level for Γ we obtain

$$\delta m^2 = g^2 \frac{\Lambda^2 \pi^2}{2(2\pi)^4} \Gamma(1 + \frac{1}{n}) \left((3\xi + 4)(1 - \frac{1}{2^{\frac{1}{n}}}) - \frac{9}{2^{1+\frac{1}{n}}} \right)$$

There are two interesting limits:

$$\xi \to 0 \qquad \delta m^2 = -g^2 \frac{\Lambda^2 \pi^2}{2(2\pi)^4} \Gamma(1 + \frac{1}{n}) \frac{1 + 8(2 - 2^{\frac{1}{n}})}{2^{1+\frac{1}{n}}}$$

$$n \to \infty \qquad \delta m^2 = -g^2 \frac{\Lambda^2 \pi^2}{2(2\pi)^4} \frac{9}{2}$$

Spin–Charge decomposition

Let $n \to \infty$, $\xi = 1$, $\{e_1, e_2\}$ - an orthonormal basis in the plane A^1_μ, A^2_μ

$$A^+_\nu = \psi_1 e_\nu + \psi_2 \bar{e}_\nu, \qquad e = e_1 + i e_2, \qquad e_a e_b = \delta_{ab},$$

$$A^-_\nu = \psi^*_2 e_\nu + \psi^*_1 \bar{e}_\nu, \quad \rho^2 = |\psi_1|^2 + |\psi_2|^2, \qquad \vec{t} = \frac{1}{\rho^2} \left(\psi^*_1, \psi^*_2\right) \vec{\sigma} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Degrees of freedom $2 \cdot 4 = (3+2) + 2 \cdot 2 - \dim U_I(1)$, $u \in U_I$, $e \mapsto ue$
Terms describing the condensate ρ:

$$A^+ A^- = \rho^2, \qquad (A^+_\mu A^-_\mu)^2 - A^+_\mu A^+_\mu A^-_\nu A^-_\nu = (\rho^2 t_3)^2$$

The ground state corresponds to $t_3 = 1$ at large spatial distances.

$$\Gamma^{\Lambda\infty} = \int d^4x \, (\partial\rho)^2 - g^2 \mu^2_\Lambda \rho^2 + \frac{g^2}{2} \rho^4 + \dots, \qquad \rho = \mu$$

$$\mathcal{L}_{eff} = (\partial\rho)^2 - \mu_\Lambda^2\rho^2 + \frac{1}{2}\rho^4 + \frac{1}{4}\left[\frac{1}{2}\mathbf{n}(\partial_\mu\mathbf{n}\times\partial_\nu\mathbf{n}) - (\partial_\mu(n_3\hat{C}_\nu) - \partial_\nu(n_3\hat{C}_\mu))\right]^2 + \frac{\rho^2}{4}(D^{\hat{C}}\mathbf{n})^2 + \frac{\rho^2}{2}\Big((\partial\mathbf{p})^2 + (\partial\mathbf{q})^2\Big) + \mathcal{L}^{FP} + \mathcal{L}_{int}$$

$$H_{\mu\nu} = \frac{i}{2}(e_\mu\bar{e}_\nu - e_\nu\bar{e}_\mu), \qquad \mathbf{p}_i = H_{0i}, \qquad \mathbf{q}_i = \frac{1}{2}\epsilon_{ijk}H_{jk}$$

$$e = e^{i\eta}\hat{e}_{\mathbf{p},\mathbf{q}}, \qquad \mathbf{n}_\pm = e^{2i\eta}t_\pm, \qquad \mathbf{n}_3 = t_3$$

$$\hat{C}_\mu = i\hat{\bar{e}}\partial_\mu\hat{e}, \qquad (D_\mu^{\hat{C}})^{ij} = \delta^{ij}\partial_\mu + 2\epsilon^{ij3}\hat{C}_\mu$$

$\mathbf{p}$, $\mathbf{q}$, n_3, ρ are invariant under $U_I(1)$, $U_C(1)$.

$$\mathcal{L}_{\text{nlsm}} = \frac{m^2}{2}(\partial\mathbf{n})^2 + \frac{1}{4}(\mathbf{n}(\partial_\mu\mathbf{n}\times\partial_\nu\mathbf{n}))^2 \quad \text{nonlinear } \sigma\text{-model}$$

Dirac Hamiltonian with a Coulomb potential

The eigenvalue problem in the spherical coordinates system $H\varphi_\lambda = \lambda\varphi_\lambda$

$$-G' + \frac{\kappa}{x}G = (z - 1 + \frac{\gamma}{x})F$$
$$F' + \frac{\kappa}{x}F = (z + 1 + \frac{\gamma}{x})G$$

$$z = \frac{\lambda}{m}, \quad \gamma = Z\alpha, \quad x = rm, \quad \kappa \in \mathbb{Z}\backslash\{0\}, \quad \varphi_\lambda(r) = \frac{1}{r}\begin{pmatrix} F_r \\ iG_r \end{pmatrix}$$

Theorem

Let $F, G \in L_\infty$. Then $\exists q > 0$ s.t. $\forall z : q^2 z^2 < z^2 - 1$ the corresponding Taylor expansions wrt the parameter $\gamma < q$ are uniformly convergent on compact sets

$$G(x) = \sum_{n=0}^{\infty} \mathsf{g}_n(x)\gamma^n, \qquad F(x) = \sum_{n=0}^{\infty} \mathsf{f}_n(x)\gamma^n$$

Hartree–Fock

$$e^{W(\boldsymbol{\eta},\bar{\boldsymbol{\eta}})} = \int d\mu_{\bar{\boldsymbol{\psi}}\boldsymbol{\psi}}\, e^{-\frac{e^2}{2}\langle Tr(\bar{\boldsymbol{\psi}}\gamma^\mu\boldsymbol{\psi})C_{\mu\nu}Tr(\bar{\boldsymbol{\psi}}\gamma^\nu\boldsymbol{\psi})+Tr(\bar{\boldsymbol{\psi}}\gamma^\mu\boldsymbol{\psi}C_{\mu\nu}\bar{\boldsymbol{\psi}}\gamma^\nu\boldsymbol{\psi})\rangle+\langle\bar{\boldsymbol{\psi}}\boldsymbol{\eta}\rangle+\langle\bar{\boldsymbol{\eta}}\boldsymbol{\psi}\rangle}$$

$\boldsymbol{\psi} = (\psi_1 \ldots \psi_N)$, $\bar{\boldsymbol{\psi}} = (\bar{\psi}_1 \ldots \bar{\psi}_N)$ where N is the number of electrons

Green's functions:

$$(\partial\!\!\!/ - ie\mathcal{A}\!\!\!/ + m)S_{xy} = \delta^4_{xy} \qquad \mathcal{A}_0 = \frac{Ze}{r} \qquad C_{\mu\nu}(t,r) = \begin{pmatrix} -\frac{\delta_t}{4\pi r} & 0 \\ 0 & 0 \end{pmatrix}$$

The semiclassical effective action gives the Hartree–Fock approximation

$$\sum_{i=1}^{N+1}\langle\bar{\phi}_i(\partial\!\!\!/ - ie\mathcal{A}\!\!\!/ + m)\phi_i\rangle + \frac{e^2}{2}\sum_{ij=1}^{N+1}\langle\bar{\phi}_i\gamma^\mu\phi_i C_{\mu\nu}\bar{\phi}_j\gamma^\nu\phi_j\rangle + \langle\bar{\phi}_i\gamma^\mu\phi_j C_{\mu\nu}\bar{\phi}_j\gamma^\nu\phi_i\rangle$$

Background field method (Large N)

Translate $\psi \to \psi + \phi$ by a field $\phi \in \mathcal{G}_{0\Lambda}$

$$e^{W^{0\infty}(\eta,\bar{\eta})} = e^{-I^{\Lambda}(\bar{\phi},\phi)+\langle\bar{\phi}\eta\rangle+\langle\bar{\eta}\phi\rangle+\tilde{W}^{0\Lambda}(\eta-\delta_{\bar{\phi}}I^{\Lambda}(\bar{\phi},\phi),\bar{\eta}+\delta_{\phi}I^{\Lambda}(\bar{\phi},\phi))}$$

$$e^{\tilde{W}^{0\Lambda}(\eta,\bar{\eta})} = \int d\mu^{0\Lambda}_{\bar{\psi}\psi} e^{-\sum_{n\geqslant 2} \mathrm{L}^{\Lambda\infty}_{n}(\bar{\psi},\psi)+\langle\bar{\eta}\psi\rangle+\langle\bar{\psi},\eta\rangle}$$

$$\mathrm{L}^{\Lambda\infty}_{n}(\bar{\psi},\psi) = \sum_{j+i=n} \frac{(\psi\delta_{\phi})^{i}\,(\bar{\psi}\delta_{\bar{\phi}})^{j}}{i!j!}\mathrm{L}^{\Lambda\infty}(\bar{\phi},\phi)$$

$$I^{\Lambda}(\bar{\phi},\phi) = \langle\bar{\phi}S^{-1}_{0\Lambda}\phi\rangle + \mathrm{L}^{\Lambda\infty}(\bar{\phi},\phi)$$

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$$\Gamma^{0\infty}(\bar{\phi},\phi) = I^{\Lambda}(\bar{\phi},\phi) + \tilde{\Gamma}^{0\Lambda}(0)$$

$$\Gamma^{0\infty}(\bar{\phi},\phi) = I^{\Lambda}(\bar{\phi},\phi) - \frac{\hbar}{2} Tr\log\left[\Gamma''^{0\infty}(\bar{\phi},\phi)\mathcal{S}_{0\Lambda}\right] + O(\hbar^{2})$$

Perturbation theory: a valence electron

$$\hat{H} = \alpha^k \partial_k - \frac{\alpha Z}{r} + \beta m - \alpha \frac{\left[\sum_{i=1}^{N-1} \varphi_i(\vec{x})\varphi_i^+(\vec{y}) + \rho(\vec{x},\vec{y})\right]}{|\vec{x}-\vec{y}|} +$$

$$+ \alpha \int_{\vec{z}\in\mathbb{R}^3} \frac{Tr\left[\sum_{i=1}^{N-1} \varphi_i(\vec{z})\varphi_i^+(\vec{z}) + \rho(\vec{z},\vec{z})\right]}{|\vec{x}-\vec{z}|}$$

Using the bilinear formula for the resolvent $(\hat{H} - \omega)\bar{g}_\omega = \delta^3$

$$\bar{g}_\omega(\vec{x},\vec{y}) = \int dz \, \frac{\varphi_z(\vec{x})\varphi_z^+(\vec{y})}{z_\mp - \omega}, \quad z_\mp = \begin{cases} z - i\epsilon & z \in \sigma_+(\hat{H}), \\ z + i\epsilon & otherwise, \end{cases} \quad \hat{H}\varphi_z = z\varphi_z$$

where $\sigma_+(\hat{H})$ is the essential and discrete spectra of unoccupied orbitals.

Multiscale expansion

Given a generating functional $\mathrm{L}_b^{\Lambda_b\infty}$ and a propagator $\mathcal{S}^b_{\Lambda_a\Lambda_b}$ we have

$$e^{-\mathrm{L}_b^{\Lambda_a\infty}(\Phi)} = \int d\mu^b_{\Lambda_a\Lambda_b} e^{-\mathrm{L}_b^{\Lambda_b\infty}(\,\cdot\,+\Phi)} = \left(\frac{\det \mathcal{S}^b_{\Lambda_a\Lambda_b}}{\det \mathcal{S}^a_{\Lambda_a\Lambda_b}}\right)^{\frac{1}{2}} \int d\mu^a_{\Lambda_a\Lambda_b} e^{-\mathrm{L}_a^{\Lambda_b\infty}(\,\cdot\,+\Phi)}$$

$$\mathrm{L}_b^{\Lambda_a\infty}(\Phi) = \mathrm{L}_a^{\Lambda_a\infty}(\Phi) - \frac{1}{2} Tr \log(\mathcal{S}^{a\;-1}_{\Lambda_a\Lambda_b} \mathcal{S}^b_{\Lambda_a\Lambda_b})$$

We get an expansion with an appropriate propagator at each scale

$$\Gamma^{0\Lambda}(0) = -\frac{\hbar}{2} \lim_{K\to\infty} \sum_{k=1}^{K} Tr \log\left[\left(\mathcal{S}^k_{\Lambda_k\Lambda_{k+1}}\right)^{-1} \mathcal{S}^{k+1}_{\Lambda_k\Lambda_{k+1}}\right] + O(\hbar^2)$$

Hypothesis 1

Only corrections to the mass and the nuclear charge, $\Gamma''(0) = -\Omega^\lambda$

$$\beta(\Omega^\lambda \varphi)_{\vec{x}} = \beta \int\limits_{\vec{z}\in\mathbb{R}^3} \mu^\lambda_{\vec{z}}\, \varphi_{\vec{x}-\vec{z}} + \alpha \int\limits_{\vec{z}\in\mathbb{R}^3} \frac{\rho^\lambda_{\vec{z}}}{|\vec{x}-\vec{z}|}\, \varphi_{\vec{x}}$$

$\mu^\lambda(r)$, $\rho^\lambda(r)$ are spherically symmetric functions

$$\hat{\rho}^\lambda(p) = \int\limits_0^\infty r^2\, \rho^\lambda(r)\, e^{-prm}, \qquad \hat{\mu}^\lambda(p) = \int\limits_0^\infty r^2 \mu^\lambda(r)\, e^{-prm}$$

At $\lambda = \Lambda$ QED at 1-loop. A low energy decomposition, $p < 1$:

$$\hat{\rho}^\lambda(p) = \rho^\lambda_1 p + \rho^\lambda_2 p^2 + \rho^\lambda_3 p^3 + \rho^\lambda_4 p^4 \log p + O(p^5)$$
$$\hat{\mu}^\lambda(p) = \mu^\lambda_0 + \mu^\lambda_1 p + \mu^\lambda_2 p^2 + \mu^\lambda_3 p^3 + \mu^\lambda_4 p^4 \log p + O(p^5)$$

Hypothesis 2

For some $M \geqslant N$ we have a set of solutions $\{\varphi_i \in \mathrm{H}^1\}_{i=1}^{M}$

$$\hat{H}\varphi_n = z_n\varphi_n, \qquad \hat{H} = \alpha^k\partial_k - \frac{Z\alpha}{r} + \beta(m + V - \Sigma)$$

$0 < \lambda \leqslant z_M$:

$$\Gamma^{\lambda\Lambda}(\bar{\psi}, \psi) = \langle\bar{\psi}\Omega^{\lambda}\psi\rangle - \frac{\langle\bar{\psi}Q\bar{\psi}\rangle + \langle\psi\bar{Q}\psi\rangle}{2} + \sum_{n>2} \mathrm{L}_n^{\lambda\Lambda}$$

$$R_1^{\lambda\Lambda} = -\sigma_{\lambda\Lambda}\sum_{n=1}^{M}\varphi_n\varphi_n^{+}$$

$z_M < \lambda \leqslant \Lambda$:

$$\Gamma^{\lambda\Lambda}(\bar{\psi}, \psi) = \langle\bar{\psi}\Omega^{\lambda}\psi\rangle + \mathrm{L}_4^{\lambda\Lambda}$$

where $R_1^{\lambda\Lambda}$ corresponds to the Green function with $Z = 1$, R.A. Swainson and G. W. F. Drake.

Non-perturbative RG equations

One consider the renormalization group equation for two point function. At the leading order we have

$$\dot{\Omega}_{xy} = -\langle \dot{\mathcal{S}}_{\lambda\Lambda} \Gamma^{\bar{\psi}\psi\bar{\psi}_x\psi_y} \rangle + O(\alpha^2)$$

By smearing the both sides with solutions of hydrogen atom of appropriate quantum numbers one obtains the necessary number of ODEs for the coefficients ρ_i^λ, μ_i^λ.