

# The renormalization group applications in QFT

- ▶ Perturbative RG:  $SU(2)$  Yang–Mills
- ▶ An effective  $SU(2)$  Yang–Mills theory at low energies
- ▶ Non-perturbative RG: Lamb shift in heavy atoms

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## $SU(2)$ Yang–Mills theory with Lorenz gauge fixing

Semiclassical Lagrangian density

$$\tilde{\mathcal{L}}_0^{tot} = \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{1}{2\xi} (\partial_\mu A_\mu^a)^2 - \partial_\mu \bar{c}^a (D_\mu c)^a,$$

$$D_\mu c = \partial_\mu c - ig[A_\mu, c],$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu],$$

$\xi > 0$  is the Feynman parameter.  $F_{\mu\nu}$ ,  $c$ ,  $\bar{c}$ ,  $A_\mu$  are elements of the algebra, e.g.  $A_\mu = t^a A_\mu^a$ ,  $(t_c)^{ab} = -i\epsilon_{abc}$ ,  $[t_a, t_b] = i\epsilon_{abc} t_c$ .

The action  $\int \tilde{\mathcal{L}}_0^{tot}$  is invariant under the transformation

$$\delta^{BRS} A_\mu = \epsilon D_\mu c, \quad \delta^{BRS} c = \epsilon \frac{1}{2} ig \{c, c\}, \quad \delta^{BRS} \bar{c} = -\epsilon \frac{1}{\xi} \partial A,$$

where  $\epsilon$  is a Grassmann parameter, and  $\{c, c\}^d = i\epsilon_{abd} c^a c^b$ .

# Regulators

$\Lambda$ ,  $\Lambda_0$  are IR and UV cutoffs,  $0 < \Lambda \leq \Lambda_0$ .

$$C_{\mu\nu}^{\Lambda\Lambda_0} = \frac{1}{p^2} (\delta_{\mu\nu} + (\xi - 1) \frac{p_\mu p_\nu}{p^2}) \sigma_{\Lambda\Lambda_0}(p^2),$$

$$S^{\Lambda\Lambda_0}(p) = \frac{1}{p^2} \sigma_{\Lambda\Lambda_0}(p^2), \quad \sigma_{\Lambda\Lambda_0}(p^2) = e^{-\frac{p^4}{\Lambda_0^4}} - e^{-\frac{p^4}{\Lambda^4}}.$$

Recover the usual covariances in the limit  $\Lambda \rightarrow 0$ ,  $\Lambda_0 \rightarrow \infty$ .

Def: a Gaussian measure  $d\tilde{\mu}(A, c, \bar{c})$  with the characteristic function:

$$e^{\frac{1}{\hbar} \langle \bar{\eta}, S^{\Lambda\Lambda_0} \eta \rangle - \frac{1}{2\hbar} \langle j, C^{\Lambda\Lambda_0} j \rangle} = \int d\tilde{\mu}_{\Lambda\Lambda_0}(\Phi) e^{\frac{i}{\hbar} \langle \Phi, K \rangle},$$

where  $\Phi = (A, c, \bar{c})$  and  $K = (j, \bar{\eta}, \eta)$ .

# The counterterms

All counterterms respecting the global symmetries and having ghost number zero

$$\begin{aligned}
 \mathcal{L}_{ct}^{\Lambda_0 \Lambda_0} = & r^{\bar{c}c\bar{c}c} \bar{c}^b c^b \bar{c}^a c^a + r_1^{\bar{c}cAA} \bar{c}^b c^b A_\mu^a A_\mu^a + r_2^{\bar{c}cAA} \bar{c}^a c^b A_\mu^a A_\mu^b \\
 & + r_1^{A^4} A_\mu^b A_\nu^b A_\mu^a A_\nu^a + r_2^{A^4} A_\nu^b A_\nu^b A_\mu^a A_\mu^a + 2\epsilon_{abc} r^{A^3} (\partial_\mu A_\nu^a) A_\mu^b A_\nu^c \\
 & - r_1^{A\bar{c}c} \epsilon_{abd} (\partial_\mu \bar{c}^a) A_\mu^b c^d - r_2^{A\bar{c}c} \epsilon_{abd} \bar{c}^a A_\mu^b \partial_\mu c^d + \Sigma^{\bar{c}c} \bar{c}^a \partial^2 c^a \\
 & - \frac{1}{2} \Sigma_T^{AA} A_\mu^a (\partial^2 \delta_{\mu\nu} - \partial_\mu \partial_\nu) A_\nu^a + \frac{1}{2\xi} \Sigma_L^{AA} (\partial_\mu A_\mu^a)^2 \\
 & + \delta m_{AA}^2 A_\mu^a A_\mu^a + \delta m_{\bar{c}c}^2 \bar{c}^a c^a.
 \end{aligned}$$

There are eleven **marginal counterterms** which depend on  $\Lambda_0$ ,  $\delta m_{\bar{c}c}^2 = 0$ .  
Expansion in powers  $\hbar$ :

$$\mathcal{L}_{ct} = \sum_{l>0} \hbar^l \mathcal{L}_{ct;l}$$

## The BRST symmetry

To obtain the antighost equation (AGE) we need an auxiliary field. Full tree level Lagrangian density in the limit  $\Lambda \rightarrow 0$ ,  $\Lambda_0 \rightarrow \infty$  is

$$\mathcal{L}_0^{tot} = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{\xi}{2} B^2 - iB \partial_\mu A_\mu - \partial_\mu \bar{c} D_\mu c.$$

The action  $\int \mathcal{L}_0^{tot}$  is invariant under the infinitesimal transformation

$$\begin{aligned} \delta^{BRST} A &= \epsilon Dc, & \delta^{BRST} c &= \epsilon \frac{1}{2} ig\{c, c\}, \\ \delta^{BRST} \bar{c} &= \epsilon iB, & \delta^{BRST} B &= 0, \end{aligned}$$

Defining the classical operator  $s$

$$\delta^{BRST} \Phi = \epsilon s \Phi, \quad \implies \quad s^2 = 0.$$

BRST invariance is explicitly **broken** by the regulators.

## Generating functionals

$$d\mu_{\Lambda\Lambda_0}(A, B, \dots) = d\nu_{\Lambda\Lambda_0}(A) d\nu_{\Lambda\Lambda_0}(B - i\frac{1}{\xi}\partial A) \dots$$

$$Z^{\Lambda\Lambda_0}(K) = \int d\mu_{\Lambda\Lambda_0}(\Phi) e^{-\frac{1}{\hbar}L^{\Lambda_0\Lambda_0}} e^{\frac{1}{\hbar}\langle K, \Phi \rangle}$$

$$\begin{aligned} \mathcal{L}^{\Lambda_0\Lambda_0} &= g\epsilon_{abc}\partial_\mu A_\nu^a A_\mu^b A_\nu^c + \frac{g^2}{4}\epsilon_{cab}\epsilon_{c ds}A_\mu^a A_\nu^b A_\mu^d A_\nu^s \\ &\quad - g\epsilon_{abc}\partial_\mu \bar{c}^a A_\mu^b c^c + \mathcal{L}_{ct}^{\Lambda_0\Lambda_0} \end{aligned}$$

The tree level interaction does not depend on the  $B$  field.

Generator of Connected Schwinger functions:

$$W^{\Lambda\Lambda_0} = \hbar \log Z^{\Lambda\Lambda_0}$$

Generator of Connected Amputated Schwinger functions:

$$L^{\Lambda\Lambda_0}(\Phi) = -\hbar \log \int d\mu_{\Lambda\Lambda_0}(\Phi') e^{-\frac{1}{\hbar}L^{\Lambda_0\Lambda_0}(\Phi'+\Phi)}$$

# The effective action

The effective action is the Legendre transform of  $W$

$$\Gamma^{\Lambda\Lambda_0}(\Phi) = \langle K, \Phi \rangle - W^{\Lambda\Lambda_0}(K).$$

Reduced effective action:

$$\Gamma^{\Lambda\Lambda_0}(\Phi) = \Gamma^{\Lambda\Lambda_0}(\Phi) - \frac{1}{2} \langle \Phi, \mathbf{C}_{\Lambda\Lambda_0}^{-1} \Phi \rangle$$

Expansion in powers  $\hbar$ :

$$\Gamma = \sum_{l=0}^{\infty} \hbar^l \Gamma_l, \quad \Gamma_{l=0}^{\Lambda\Lambda_0; \phi\phi} = 0$$

Here  $\delta\left(\sum_{i=0}^{n-1} p_i\right) \Gamma^{\vec{\phi}; w}(\vec{p}) := \partial^w \left( \prod_{i=0}^{n-1} \frac{\delta}{\delta\phi_i(p_i)} \right) \Gamma \Big|_{\vec{\phi}=0}$ ,  $w = (0, w_1, \dots, w_{n-1})$ .

## Main idea

The characteristic function is the Fourier transform of the measure:

$$\begin{aligned}\chi(j) &= \mathcal{N} \int [d\phi] e^{-\frac{1}{2\hbar} \langle \phi C_{\Lambda\Lambda_0}^{-1} \phi \rangle + i\frac{1}{\hbar} \langle j\phi \rangle} \\ &= \int d\mu(\phi) e^{i\frac{1}{\hbar} \langle j\phi \rangle} = e^{-\frac{1}{2\hbar} \langle j C_{\Lambda\Lambda_0} j \rangle}\end{aligned}$$

It is used to obtain all moments:

$$E(\phi(x_0), \dots, \phi(x_{N-1})) = \prod_{i=0}^{N-1} \frac{\hbar}{i} \frac{\delta}{\delta j(x_i)} \chi(j) \Big|_{j=0}$$

Important differential equation:

$$\begin{aligned}\frac{d}{d\Lambda} \chi(j) &= \dot{\chi}(j) = -\frac{1}{2\hbar} \langle j \dot{C}_{\Lambda\Lambda_0} j \rangle e^{\frac{1}{2\hbar} \langle j C_{\Lambda\Lambda_0} j \rangle} \\ &= \frac{\hbar}{2} \int d\mu(\phi) \left\langle \frac{\delta}{\delta \phi} \dot{C}_{\Lambda\Lambda_0} \frac{\delta}{\delta \phi} \right\rangle e^{i\frac{1}{\hbar} \langle j\phi \rangle}\end{aligned}$$



## RG equation

Wilson; Wegner; Polchinski...

$$\dot{L} = \frac{\hbar}{2} \left\langle \frac{\delta}{\delta\Phi}, \dot{\mathbf{C}} \frac{\delta}{\delta\Phi} \right\rangle L - \frac{1}{2} \left\langle \frac{\delta L}{\delta\Phi}, \dot{\mathbf{C}} \frac{\delta L}{\delta\Phi} \right\rangle \quad \dot{L} := \partial_\Lambda L^{\Lambda\Lambda_0}$$

Wetterich; Morris; Bonini, D'Attanasio, Marchesini...

$$\begin{aligned} \dot{\Gamma} &= \frac{\hbar}{2} \langle \dot{\mathbf{C}}, L'' \rangle & L'' &= \Gamma'' (1 + \hat{\mathbf{1}} \mathbf{C} \Gamma'')^{-1} \\ \dot{\Gamma} &= \frac{\hbar}{2} \langle \dot{\mathbf{C}} \Gamma'' \sum_{m=0}^{\infty} (-\hat{\mathbf{1}} \mathbf{C} \Gamma'')^m \rangle & \hat{\mathbf{1}} &= \begin{cases} \delta_{\varphi^a \varphi^b} \\ -\delta_{c_a c_b} & -\delta_{\bar{c}_a \bar{c}_b} \end{cases} \end{aligned}$$

## Trivial dependence on the $B$ field (linear gauge)

Vanishing renormalization conditions for all relevant terms,

$$\partial^w \Gamma_l^{0\Lambda_0; B\vec{\phi}}(\vec{q}) = 0,$$

imply that  $\forall \Lambda > 0$

$$\Gamma_l^{\Lambda\Lambda_0; B\vec{\phi}}(\vec{p}) = 0.$$

Thus there are no counterterms with the field  $B$ .

Dependence on the  $B$ -fields known explicitly:

$$\Gamma^{\Lambda\Lambda_0}(A, B, c, \bar{c}) = \frac{1}{2\xi} \int d^4x (\xi B - i\partial A)^2 + \tilde{\Gamma}^{\Lambda\Lambda_0}(A, c, \bar{c}).$$

where  $\tilde{\Gamma}^{\Lambda\Lambda_0}$  is defined using the measure  $d\tilde{\mu}(A, c, \bar{c})$ .

# Violated Slavnov–Taylor Identities (VSTI)

Introduce the Lagrangian density

$$\mathcal{L}_{vst}^{\Lambda_0\Lambda_0} = \mathcal{L}^{\Lambda_0\Lambda_0} + \gamma\psi^{\Lambda_0} + \omega\Omega^{\Lambda_0},$$

where  $\gamma, \omega$  are external sources,  $R_i^{\Lambda_0} = 1 + O(\hbar)$ ,

$$\psi^{\Lambda_0} = R_1^{\Lambda_0}\partial c - igR_2^{\Lambda_0}[A, c], \quad \Omega^{\Lambda_0} = \frac{1}{2i}gR_3^{\Lambda_0}\{c, c\}.$$

We consider the change of variables  $\Phi \mapsto \Phi + \delta_\epsilon \Phi$

$$\delta_\epsilon A = \epsilon \sigma_{0\Lambda_0} \psi^{\Lambda_0}, \quad \delta_\epsilon c = -\epsilon \sigma_{0\Lambda_0} \Omega^{\Lambda_0}, \quad \delta_\epsilon \bar{c} = \epsilon \sigma_{0\Lambda_0} iB,$$

which leaves invariant the functional integral:  $\delta_\epsilon Z^{0\Lambda_0} = 0$ , with

$$Z^{0\Lambda_0}(K) = \int d\mu_{0\Lambda_0}(\Phi) e^{-\frac{1}{\hbar}L_{vst}^{\Lambda_0\Lambda_0}} e^{\frac{1}{\hbar}\langle K, \Phi \rangle}.$$

Performing the Legendre transform the identity  $\delta_\epsilon Z^{0\Lambda_0} = 0$  gives

$$\Gamma_1^{0\Lambda_0} + \int d^4x (iB + \frac{1}{\xi} \partial A) \Gamma_\beta^{0\Lambda_0} = \frac{1}{2} \mathcal{S} \underline{\Gamma}^{0\Lambda_0},$$

where we have introduced an auxiliary functional  $\underline{\Gamma}^{0\Lambda_0}$

$$\underline{\Gamma}^{0\Lambda_0} = i\langle B, \bar{\omega} \rangle + \tilde{\underline{\Gamma}}^{0\Lambda_0}, \quad \tilde{\underline{\Gamma}}^{0\Lambda_0} = \tilde{\Gamma}^{0\Lambda_0} + \frac{1}{2\xi} \langle A, \partial\partial A \rangle,$$

and  $\mathcal{S} = \mathcal{S}_{\tilde{c}} + \mathcal{S}_A - \mathcal{S}_c$ , with

$$\mathcal{S}_\phi = \left\langle \frac{\delta \underline{\Gamma}^{0\Lambda_0}}{\delta \phi}, \sigma_{0\Lambda_0} \frac{\delta}{\delta \phi^\dagger} \right\rangle + \left\langle \frac{\delta \tilde{\underline{\Gamma}}^{0\Lambda_0}}{\delta \phi^\dagger}, \sigma_{0\Lambda_0} \frac{\delta}{\delta \phi} \right\rangle,$$

$$(\phi, \phi^\dagger) \in \{(A, \gamma), (c, \omega), (\tilde{c}, \bar{\omega})\}, \quad \frac{\delta}{\delta \tilde{c}} = \frac{\delta}{\delta \bar{c}} - \partial \frac{\delta}{\delta \gamma}.$$

## A variation on nilpotency

We rewrite the VSTI in the following form

$$\langle iB, \tilde{\Gamma}_\beta^{0\Lambda_0} \rangle = \frac{1}{2} \mathcal{S}_{\tilde{c}} \Gamma^{0\Lambda_0} = \langle iB, \sigma_{0\Lambda_0} \frac{\delta}{\delta \tilde{c}} \tilde{\Gamma}^{0\Lambda_0} \rangle,$$
$$\tilde{\mathbf{F}}_1^{0\Lambda_0} = \frac{1}{2} \tilde{\mathcal{S}} \tilde{\Gamma}^{0\Lambda_0},$$

where

$$\tilde{\mathcal{S}} = \mathcal{S}_A - \mathcal{S}_c, \quad \tilde{\mathbf{F}}_1^{0\Lambda_0} = \tilde{\Gamma}_1^{0\Lambda_0} + \frac{1}{\xi} \langle \partial A, \tilde{\Gamma}_\beta^{0\Lambda_0} \rangle.$$

Important properties of  $\mathcal{S}_\phi$ :  $\forall \phi, \phi' \in \{A, c, \tilde{c}\}$

$$(\mathcal{S}_\phi \mathcal{S}_{\phi'} + \mathcal{S}_{\phi'} \mathcal{S}_\phi) \Gamma^{0\Lambda_0} = 0.$$

It follows that  $\tilde{\mathcal{S}}^2 \tilde{\Gamma}^{0\Lambda_0} = 0$ ,  $\tilde{\mathcal{S}} \mathcal{S}_{\tilde{c}} = -\mathcal{S}_{\tilde{c}} \tilde{\mathcal{S}}$  and consequently

$$\tilde{\mathcal{S}} \tilde{\mathbf{F}}_1 = 0, \quad \tilde{\mathcal{S}} \tilde{\Gamma}_\beta + \sigma_{0\Lambda_0} \left( \frac{\delta}{\delta \tilde{c}} - \partial \frac{\delta}{\delta \gamma} \right) \tilde{\mathbf{F}}_1 = 0.$$

## Explicit form of the VSTI

$$\tilde{\Gamma}_\beta^{0\Lambda_0} = \sigma_{0\Lambda_0} \left( \frac{\delta \tilde{\Gamma}^{0\Lambda_0}}{\delta \bar{c}} - \partial \tilde{\Gamma}_\gamma^{0\Lambda_0} \right) \quad (AGE)$$

$$\tilde{\mathbf{F}}_1^{0\Lambda_0} = \left\langle \frac{\delta \tilde{\Gamma}^{0\Lambda_0}}{\delta A}, \sigma_{0\Lambda_0} \tilde{\Gamma}_\gamma^{0\Lambda_0} \right\rangle - \left\langle \frac{\delta \tilde{\Gamma}^{0\Lambda_0}}{\delta c}, \sigma_{0\Lambda_0} \tilde{\Gamma}_\omega^{0\Lambda_0} \right\rangle \quad (STI)$$

The goal is to show that  $\tilde{\Gamma}_\beta^{0\infty} = 0$ ,  $\tilde{\mathbf{F}}_1^{0\infty} = 0$  which imply  $\tilde{\Gamma}_1^{0\infty} = 0$ .

Another form of the VSTI

$$\begin{aligned} \tilde{\Gamma}_1^{0\Lambda_0} &= \left\langle \frac{\delta \tilde{\Gamma}^{0\Lambda_0}}{\delta A}, \sigma_{0\Lambda_0} \tilde{\Gamma}_\gamma^{0\Lambda_0} \right\rangle - \left\langle \frac{\delta \tilde{\Gamma}^{0\Lambda_0}}{\delta c}, \sigma_{0\Lambda_0} \tilde{\Gamma}_\omega^{0\Lambda_0} \right\rangle \\ &\quad - \frac{1}{\xi} \left\langle \partial A, \sigma_{0\Lambda_0} \frac{\delta \tilde{\Gamma}^{0\Lambda_0}}{\delta \bar{c}} \right\rangle. \end{aligned}$$

The vertex functions on the rhs give renormalization conditions for the anomalies on the lhs.

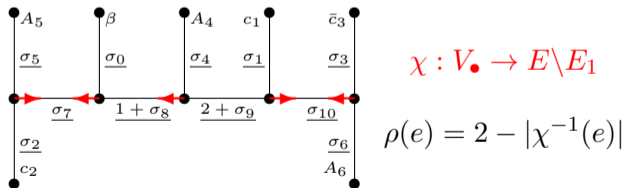
## Renormalization conditions

No tilde on  $\Gamma(A, c, \bar{c})$

- ▶  $\partial^w \Gamma_{\vec{z}}^{0\Lambda_0; \vec{\phi}}(0) = 0$ , for all strictly relevant terms,  $\varkappa \in \{\gamma, \omega\}$ .
- ▶  $\Gamma^{M\Lambda_0; c\bar{c}\bar{c}}(0) = 0$ ,  $\Gamma^{M\Lambda_0; c\bar{c}A^2}(0) = 0$ ,  $\partial_A \Gamma^{M\Lambda_0; c\bar{c}A}(0) = 0$ . We show that the counterterms  $r^{\bar{c}c\bar{c}}$ ,  $r_1^{\bar{c}cA^2}$ ,  $r_2^{\bar{c}cA^2}$ ,  $r_2^{A\bar{c}c}$  vanish.
- ▶ The renormalization constants  $r^{A^3}$ ,  $\Sigma_T^{AA}$ ,  $\Sigma^{\bar{c}c}$  are free.
- ▶ The remaining 7 renormalization constants must satisfy 7 additional relations in order to make the marginal terms  $\Gamma_1^{\vec{\phi}; w}$ ,  $\Gamma_\beta^{\vec{\phi}; w}$  at the renormalization point comply with the bounds **3** and **4** below.
- ▶ We prove the existence of a solution for this system of relations that does not depend on the UV cutoff.

## Trees $\mathcal{T}_{\vec{\phi}}$ and $\mathcal{T}_{1\vec{\phi}}$

- ▶ Vertices of valence one and three,  $V = V_1 \cup V_3$ .
- ▶ An edge  $e$  carries momentum  $p_e$ . Momentum conservation at the vertices.
- ▶ To any edge are associated  $\rho(e) \in \{0, 1, 2\}$  and the number of momentum derivatives  $\sigma(e)$ .
- ▶ An edge has a  $\theta$ -weight,  $\theta(e) = \rho(e) + \sigma(e)$ .



$$\forall \tau \in \mathcal{T}_{\phi_0, \dots, \phi_{n-1}}, n \geq 4: \theta(\tau) = n + \|w\| - 4, \theta(\tau) = -[\Gamma \vec{\phi}; w]$$



## Main elements of the bounds

For a tree  $\tau$  we sum over the family of  $\theta$ -weights  $\Theta_\tau^w = \{\theta_j(e)\}_j$ .

$$\begin{aligned} \Pi_{\tau,\theta}^\Lambda(\vec{p}) &= \prod_{e \in E} \frac{1}{(\Lambda + |p_e|)^{\theta(e)}}, \\ Q_\tau^{\Lambda;w}(\vec{p}) &= \begin{cases} \sum_{\Theta_\tau^w} \Pi_{\tau,\theta}^\Lambda(\vec{p}), & |V_1| > 3, \\ \inf_{i \in \mathbb{I}} \sum_{\Theta_\tau^{w'(i)}} \Pi_{\tau,\theta}^\Lambda(\vec{p}), & |V_1| = 3. \end{cases} \end{aligned}$$

$w'(i)$  is obtained from  $w$  by diminishing  $w_i$  by one unit.

For nonvanishing  $w$   $\mathbb{I} = \{i : w_i > 0\}$ .

$$\eta(\vec{p}) = \min_{\mathbb{I} \in \wp_{n-1} \setminus \{\emptyset\}} \left( \left| \sum_{i \in \mathbb{I}} p_i \right|, M \right)$$

## Bounds on vertex functions

Let  $d = 4 - 2n_{\mathcal{X}} - N - \|w\|$ .

**1.a)**  $d \geq 0$  or  $N + n_{\mathcal{X}} = 2$

$$|\Gamma_{\vec{z};l}^{\Lambda\Lambda_0;\vec{\phi};w}(\vec{p})| \leq (\Lambda + |\vec{p}|)^d P_r^{\Lambda\Lambda}(\vec{p}),$$

**1.b)**  $d < 0$

$$|\Gamma_{\vec{z};l}^{\Lambda\Lambda_0;\vec{\phi};w}(\vec{p})| \leq \sum_{\tau \in \mathcal{T}_{\vec{z}\vec{\phi}}} Q_{\tau}^{\Lambda;w}(\vec{p}) P_r^{\Lambda\Lambda}(\vec{p}),$$

$$P_r^{\Lambda\Lambda'}(\vec{p}) = \mathcal{P}_r^{(0)}\left(\log_+ \frac{\max(|\vec{p}|, M)}{\Lambda + \eta(\vec{p})}\right) + \mathcal{P}_r^{(1)}\left(\log_+ \frac{\Lambda'}{M}\right).$$

$\mathcal{P}_r$  denotes polynomials with nonnegative coefficients and degree

$$r = \begin{cases} 2l & d \geq 0 \\ 2l - 1 & d < 0 \end{cases}$$

## UV convergence

Let  $d = 4 - 2n_\varkappa - N - \|w\|$ .

**2.a)**  $d \geq 0$  or  $N + n_\varkappa = 2$

$$|\partial_{\Lambda_0} \Gamma_{\vec{z};l}^{\Lambda\Lambda_0;\vec{\phi};w}(\vec{p})| \leq \frac{\Lambda + M + |\vec{p}|}{\Lambda_0^2} (\Lambda + |p|)^d P_r^{\Lambda\Lambda_0}(\vec{p}),$$

**2.b)**  $d < 0$

$$|\partial_{\Lambda_0} \Gamma_{\vec{z};l}^{\Lambda\Lambda_0;\vec{\phi};w}(\vec{p})| \leq \frac{\Lambda + M + |\vec{p}|}{\Lambda_0^2} \sum_{\tau \in \mathcal{T}_{\vec{z}\vec{\phi}}} Q_\tau^{\Lambda;w}(\vec{p}) P_r^{\Lambda\Lambda_0}(\vec{p}).$$

Cauchy criterion

$$|\Gamma_{\vec{z};l}^{\Lambda\Lambda'_0;\vec{\phi};w} - \Gamma_{\vec{z};l}^{\Lambda\Lambda_0;\vec{\phi};w}| \leq \int_{\Lambda_0}^{\Lambda'_0} d\lambda_0 |\partial_{\lambda_0} \Gamma_{\vec{z};l}^{\Lambda\lambda_0;\vec{\phi};w}|.$$

## Restoration of AGE

Let  $d = 3 - 2n_{\varkappa} - N - \|w\|$ .

**3.a)**  $d \geq 0$  or  $N + n_{\varkappa} = 1$

$$|\Gamma_{\beta \vec{z}; l}^{\Lambda \Lambda_0; \vec{\phi}; w}(\vec{p})| \leq \frac{M + |\vec{p}| + \Lambda}{\Lambda_0} (\Lambda + |\vec{p}|)^d P_{r0}^{\Lambda \Lambda_0}(\vec{p}),$$

**3.b)**  $d < 0$

$$|\Gamma_{\beta \vec{z}; l}^{\Lambda \Lambda_0; \vec{\phi}; w}(\vec{p})| \leq \frac{M + |\vec{p}| + \Lambda}{\Lambda_0} \sum_{\tau \in \mathcal{T}_{\beta \vec{z} \vec{\phi}}} Q_{\tau}^{\Lambda; w}(\vec{p}) P_{r0}^{\Lambda \Lambda_0}(\vec{p}),$$

$$P_{rs}^{\Lambda \Lambda_0}(\vec{p}) = \left(1 + \left(\frac{|\vec{p}|}{\Lambda_0}\right)^4\right) \mathcal{P}_s^{(2)}\left(\frac{|\vec{p}|}{\Lambda + M}\right) P_r^{\Lambda \Lambda_0}(\vec{p}).$$

# Restoration of STI

Let  $d = 5 - 2n_{\mathcal{X}} - N - \|w\|$ .

**4.a)**  $d > 0$  or  $N + n_{\mathcal{X}} = 2$

$$|\Gamma_{1\vec{z};l}^{\Lambda\Lambda_0;\vec{\phi};w}(\vec{p})| \leq \frac{M + |\vec{p}| + \Lambda}{\Lambda_0} (\Lambda + |\vec{p}|)^d P_{r_1 s_1}^{\Lambda\Lambda_0}(\vec{p}),$$

**4.b)**  $d \leq 0$

$$|\Gamma_{1\vec{z};l}^{\Lambda\Lambda_0;\vec{\phi};w}(\vec{p})| \leq \frac{M + |\vec{p}| + \Lambda}{\Lambda_0} \sum_{\tau \in \mathcal{T}_{1\vec{z}\vec{\phi}}} Q_{\tau}^{\Lambda;w}(\vec{p}) P_{r_1 s_1}^{\Lambda\Lambda_0}(\vec{p}).$$

$r_1, s_1$  are linear functions of loop number  $l$ .

## Overview of the proof

We proceed by induction in the loop order  $l$ :  $\Gamma = \sum_{l=0}^{\infty} \hbar^l \Gamma_l$ .

The irrelevant terms are constructed by integrating the FE from  $\Lambda_0$  down to  $\Lambda$ :

$$\Gamma_l^{\Lambda\Lambda_0; \vec{\phi}; w} = \Gamma_l^{\Lambda_0\Lambda_0; \vec{\phi}; w} + \int_{\Lambda_0}^{\Lambda} d\lambda \dot{\Gamma}_l^{\lambda\Lambda_0; \vec{\phi}; w}, \quad \Gamma_l^{\Lambda_0\Lambda_0; \vec{\phi}; w} = 0$$

$$\dot{\Gamma}^{\vec{\phi}} = \frac{\hbar}{2} \sum_{\vec{\phi}_1, \dots, \vec{\phi}_m} \langle \dot{\mathbf{C}}_{\zeta_1 \bar{\zeta}_m} \Gamma^{\zeta_1 \vec{\phi}_1 \bar{\zeta}_1} \prod_{j=2}^m \mathbf{C}_{\zeta_j \bar{\zeta}_{j-1}} \Gamma^{\zeta_j \vec{\phi}_j \bar{\zeta}_j} \rangle$$

Here  $\Gamma_{l=0}^{\zeta \bar{\zeta}} = 0$ .

For the relevant terms we integrate the FE from  $\Lambda = 0$  up to arbitrary  $\Lambda$  at the renormalization point  $\vec{q}$

$$\Gamma_l^{\Lambda\Lambda_0;\vec{\phi};w}(\vec{q}) = \Gamma_l^{0\Lambda_0;\vec{\phi};w}(\vec{q}) + \int_0^\Lambda d\lambda \dot{\Gamma}_l^{\lambda\Lambda_0;\vec{\phi};w}(\vec{q})$$

We interpolate this to arbitrary momenta  $\vec{p}$  integrating over the corresponding irrelevant terms;

$$\Gamma_l^{\Lambda\Lambda_0;\vec{\phi};w}(\vec{p}) = \Gamma_l^{\Lambda\Lambda_0;\vec{\phi};w}(\vec{q}) + \int_{\vec{q}}^{\vec{p}} \partial \Gamma_l^{\Lambda\Lambda_0;\vec{\phi};w}$$

## Renormalization conditions on $\Gamma_1$

$$\Gamma_1^{cAAA} \sim \Gamma^{AAAA} \Gamma_\gamma^c + \sum_{Z_3} \Gamma^{AAA} \Gamma_\gamma^{Ac} + \Gamma^{AA} \Gamma_\gamma^{cAA}$$

$\sim$  stands for equality up to terms  $\frac{M}{\Lambda_0} \log^{r_1} \frac{\Lambda_0}{M}$ .

5 equations  $u_i^{cAAA}$  and 2 variables  $R_1^{AAAA}, R_2^{AAAA}$

$$u_{1,2}^{cAAA} = 8R_1 R_2^{AAAA} - 4g R_2 R^{AAA} + \zeta_{1,2}^{\Lambda_0} \sim 0$$

$$u_{3,4,5}^{cAAA} = 4R_1 R_1^{AAAA} + 2g R_2 R^{AAA} + \zeta_{3,4,5}^{\Lambda_0} \sim 0$$

Using AGE  $\Gamma_\beta \sim 0$

$$\begin{aligned} \tilde{S}\Gamma_1 \sim 0 \\ \Gamma_1^{\bar{c}} - \partial\Gamma_{1;\gamma} \sim 0 \end{aligned} \implies \begin{aligned} u_1^{cAAA} &\sim u_2^{cAAA} \\ u_3^{cAAA} &\sim u_4^{cAAA} \sim u_5^{cAAA} \end{aligned}$$



## Maximal Abelian Gauge (MAG)

Extremum of  $(A^1)^2 + (A^2)^2$  over the gauge transformations.

$$A^\pm = \frac{A^2 \pm iA^1}{\sqrt{2}}, \quad A = A^3$$

Gauge transformation has the following form:

$$\begin{aligned} \delta A &= \partial\alpha - ig(A^+\alpha^- - A^-\alpha^+), & D_\mu^\pm A_\nu^\pm &= \partial_\mu A_\nu^\pm \mp igA_\mu A_\nu^\pm \\ \delta A^\pm &= D^\pm\alpha^\pm \pm ig\alpha A^\pm \end{aligned}$$

extremum  $A^+A^-$  under  $\alpha^\pm \implies D_\nu^\pm A_\nu^\pm = 0$

We fix  $U(1)$  symmetry by  $\partial A = 0$ . The t'Hooft gauge fixing Lagrangian density is

$$\mathcal{L}^{GF} = \frac{1}{2\xi}(\partial A)^2 + \frac{1}{\xi}(D_\nu^+ A_\nu^+)(D_\mu^- A_\mu^-)$$

With the  $B^a$  field we have  $\mathcal{L}^{GF} = \mathcal{L}_2^{GF} + \mathcal{L}_3^{GF}$  where

$$\mathcal{L}_2^{GF} = \frac{\xi}{2} B^2 + \xi B^+ B^- - i B^a \partial_\nu A_\nu^a, \quad a \in (+, -, 3)$$

$$\mathcal{L}_3^{GF} = g B^+ A_\nu A_\nu^- - g B^- A_\nu A_\nu^+, \quad B = B^3$$

$L^{YM} + L^{FP} + L^{GF}$  is BRST invariant, i.e.  $\delta\Phi = \theta s\Phi$

$$\begin{aligned} sA &= \partial c - ig(A^+ c_+ - A^- c_-), & sc &= igc_+ c_-, & s\bar{c}_a &= iB^a \\ sA^\pm &= D^\pm c_\pm \pm igcA^\pm, & sc_\pm &= \pm igc c_\pm, & sB^a &= 0 \end{aligned}$$

Regularization:  $\sigma_\lambda(p^2) = e^{-\left(\frac{p^2}{\lambda^2}\right)^n}$ ; BRST insertions:

$$\begin{aligned} \psi &= R_1^{\Lambda_0} \partial c + igR_2^{\Lambda_0} (A^+ c_- - A^- c_+), & \Omega &= igR_3 c_- c_+ \\ \psi^\pm &= R_4^{\Lambda_0} \partial c_\pm \mp igR_5^{\Lambda_0} A c_\pm \pm igR_6^{\Lambda_0} c A^\pm, & \Omega^\pm &= \pm igR_7 c_\pm c \end{aligned}$$

$$\lim_{\Lambda_0 \rightarrow \infty} \left( \sigma_{0\Lambda_0} \frac{\delta F}{\delta \bar{c}} - \partial \Gamma_\gamma \right) = 0 \quad (AGE)$$

$$\begin{aligned} \lim_{\Lambda_0 \rightarrow \infty} \left( \left\langle \frac{\delta F}{\delta A^a} \sigma_{0\Lambda_0} \Gamma_{\gamma^{\bar{a}}} \right\rangle - \left\langle \frac{\delta F}{\delta c_a} \sigma_{0\Lambda_0} \Gamma_{\omega^{\bar{a}}} \right\rangle \right. \\ \left. - \frac{1}{\xi} \left\langle \partial A \sigma_{0\Lambda_0} \frac{\delta F}{\delta \bar{c}} \right\rangle + i \left\langle B^\pm \sigma_{0\Lambda_0} \frac{\delta F}{\delta \bar{c}_\pm} \right\rangle + \hbar \Delta^{\Lambda\Lambda_0} \right) = 0 \quad (STI) \end{aligned}$$

$$F^{\Lambda\Lambda_0} = \frac{1}{2} \langle \Phi \mathbf{C}_{0\Lambda_0}^{-1} \Phi \rangle + \Gamma^{\Lambda\Lambda_0},$$

$$\begin{aligned} \Delta^{\Lambda\Lambda_0} = & - \langle \sigma_\Lambda (1 + \Gamma'' \hat{\mathbf{C}})_{A^a \Phi}^{-1} \frac{\delta \Gamma_{\gamma^{\bar{a}}}}{\delta \Phi} \rangle - \langle \sigma_\Lambda \partial (1 + \Gamma'' \hat{\mathbf{C}})_{A^a c_a}^{-1} \rangle \\ & + \langle \sigma_\Lambda (1 + \Gamma'' \hat{\mathbf{C}})_{c_a \Phi}^{-1} \frac{\delta \Gamma_{\omega^{\bar{a}}}}{\delta \Phi} \rangle, \end{aligned}$$

$$\Gamma'' = \frac{\delta^2 \Gamma}{\delta \Phi \delta \Phi'}, \quad \hat{\mathbf{C}} = \hat{\mathbf{1}} \mathbf{C}, \quad \hat{\mathbf{1}} = \begin{cases} \delta_{\varphi^a \varphi^b} \\ -\delta_{c_a c_b} & -\delta_{\bar{c}_a \bar{c}_b} \end{cases}$$

## Mass term of the condensate

$$\Gamma^{\Lambda\Lambda_0} = \langle A_\mu^+ (p^2 \delta_{\mu\nu} - p_\mu p_\nu) \Sigma^{\Lambda\Lambda_0} A_\nu^- \rangle + \langle A^+ \delta m_{\Lambda\Lambda_0}^2 A^- \rangle + \dots$$

Consider the vertex function  $\Gamma_1^{A^+ c^-}$ . Using VSTI we have

$$F^{A_\beta^+ A^-} \partial_{p_\alpha} \Gamma_{\gamma^+}^{c^-} + \partial_{p_\alpha} \Delta^{A_\beta^+ c^-}(p) \Big|_{p=0} = \delta_{\beta\alpha} \delta m^2 R_4 + i \frac{\partial \Delta^{A_\beta^+ c^-}(p)}{\partial p_\alpha} \Big|_{p=0} = 0$$

Substituting the tree level for  $\Gamma$  we obtain

$$\delta m^2 = g^2 \frac{\Lambda^2 \pi^2}{2(2\pi)^4} \Gamma\left(1 + \frac{1}{n}\right) \left( (3\xi + 4) \left(1 - \frac{1}{2^{\frac{1}{n}}}\right) - \frac{9}{2^{1+\frac{1}{n}}} \right)$$

There are two interesting limits:

$$\xi \rightarrow 0 \quad \delta m^2 = -g^2 \frac{\Lambda^2 \pi^2}{2(2\pi)^4} \Gamma\left(1 + \frac{1}{n}\right) \frac{1 + 8(2 - 2^{\frac{1}{n}})}{2^{1+\frac{1}{n}}}$$

$$n \rightarrow \infty \quad \delta m^2 = -g^2 \frac{\Lambda^2 \pi^2}{2(2\pi)^4} \frac{9}{2}$$

## Spin-Charge decomposition

Let  $n \rightarrow \infty$ ,  $\xi = 1$ ,  $\{e_1, e_2\}$  - an orthonormal basis in the plane  $A_\mu^1, A_\mu^2$

$$A_\nu^+ = \psi_1 e_\nu + \psi_2 \bar{e}_\nu, \quad e = e_1 + ie_2, \quad e_a e_b = \delta_{ab},$$

$$A_\nu^- = \psi_2^* e_\nu + \psi_1^* \bar{e}_\nu, \quad \rho^2 = |\psi_1|^2 + |\psi_2|^2, \quad \vec{t} = \frac{1}{\rho^2} (\psi_1^*, \psi_2^*) \vec{\sigma} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Degrees of freedom  $2 \cdot 4 = (3 + 2) + 2 \cdot 2 - \dim U_I(1)$ ,  $u \in U_I$ ,  $e \mapsto ue$

Terms describing the condensate  $\rho$ :

$$A^+ A^- = \rho^2, \quad (A_\mu^+ A_\mu^-)^2 - A_\mu^+ A_\mu^+ A_\nu^- A_\nu^- = (\rho^2 t_3)^2$$

The ground state corresponds to  $t_3 = 1$  at large spatial distances.

$$\Gamma^{\Lambda\infty} = \int d^4x (\partial\rho)^2 - g^2 \mu_\Lambda^2 \rho^2 + \frac{g^2}{2} \rho^4 + \dots, \quad \rho = \mu$$

$$\mathcal{L}_{eff} = (\partial\rho)^2 - \mu_\Lambda^2 \rho^2 + \frac{1}{2}\rho^4 + \frac{1}{4} \left[ \frac{1}{2} \mathbf{n}(\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n}) - (\partial_\mu(n_3 \hat{C}_\nu) - \partial_\nu(n_3 \hat{C}_\mu)) \right]^2 + \frac{\rho^2}{4} (D\hat{C}\mathbf{n})^2 + \frac{\rho^2}{2} \left( (\partial\mathbf{p})^2 + (\partial\mathbf{q})^2 \right) + \mathcal{L}^{FP} + \mathcal{L}_{int}$$

$$H_{\mu\nu} = \frac{i}{2}(e_\mu \bar{e}_\nu - e_\nu \bar{e}_\mu), \quad \mathbf{p}_i = H_{0i}, \quad \mathbf{q}_i = \frac{1}{2}\epsilon_{ijk} H_{jk}$$

$$e = e^{i\eta} \hat{e}_{\mathbf{p},\mathbf{q}}, \quad \mathbf{n}_\pm = e^{2i\eta} t_\pm, \quad \mathbf{n}_3 = t_3$$

$$\hat{C}_\mu = i\hat{e}\partial_\mu \hat{e}, \quad (D_\mu^{\hat{C}})^{ij} = \delta^{ij}\partial_\mu + 2\epsilon^{ij3}\hat{C}_\mu$$

$\mathbf{p}$ ,  $\mathbf{q}$ ,  $n_3$ ,  $\rho$  are invariant under  $U_I(1)$ ,  $U_C(1)$ .

$$\mathcal{L}_{\text{nlsm}} = \frac{m^2}{2}(\partial\mathbf{n})^2 + \frac{1}{4}(\mathbf{n}(\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n}))^2 \quad \text{nonlinear } \sigma\text{-model}$$

## Dirac Hamiltonian with a Coulomb potential

The eigenvalue problem in the spherical coordinates system  $H\varphi_\lambda = \lambda\varphi_\lambda$

$$-G' + \frac{\kappa}{x}G = (z - 1 + \frac{\gamma}{x})F$$

$$F' + \frac{\kappa}{x}F = (z + 1 + \frac{\gamma}{x})G$$

$$z = \frac{\lambda}{m}, \quad \gamma = Z\alpha, \quad x = rm, \quad \kappa \in \mathbb{Z} \setminus \{0\}, \quad \varphi_\lambda(r) = \frac{1}{r} \begin{pmatrix} F_r \\ iG_r \end{pmatrix}$$

### Theorem

Let  $F, G \in L_\infty$ . Then  $\exists q > 0$  s.t.  $\forall z : q^2 z^2 < z^2 - 1$  the corresponding Taylor expansions wrt the parameter  $\gamma < q$  are uniformly convergent on compact sets

$$G(x) = \sum_{n=0}^{\infty} \mathfrak{g}_n(x) \gamma^n,$$

$$F(x) = \sum_{n=0}^{\infty} \mathfrak{f}_n(x) \gamma^n$$

# Hartree–Fock

$$e^{W(\eta, \bar{\eta})} = \int d\mu_{\bar{\psi}\psi} e^{-\frac{e^2}{2} \langle \text{Tr}(\bar{\psi}\gamma^\mu\psi)C_{\mu\nu}\text{Tr}(\bar{\psi}\gamma^\nu\psi) + \text{Tr}(\bar{\psi}\gamma^\mu\psi C_{\mu\nu}\bar{\psi}\gamma^\nu\psi) \rangle + \langle \bar{\psi}\eta \rangle + \langle \bar{\eta}\psi \rangle}$$

$\psi = (\psi_1 \dots \psi_N)$ ,  $\bar{\psi} = (\bar{\psi}_1 \dots \bar{\psi}_N)$  where  $N$  is the number of electrons

Green's functions:

$$(\not{\partial} - ie\mathcal{A} + m)S_{xy} = \delta_{xy}^4 \quad \mathcal{A}_0 = \frac{Ze}{r} \quad C_{\mu\nu}(t, r) = \begin{pmatrix} -\frac{\delta_t}{4\pi r} & 0 \\ 0 & 0 \end{pmatrix}$$

The semiclassical effective action gives the Hartree–Fock approximation

$$\sum_{i=1}^{N+1} \langle \bar{\phi}_i (\not{\partial} - ie\mathcal{A} + m) \phi_i \rangle + \frac{e^2}{2} \sum_{ij=1}^{N+1} \langle \bar{\phi}_i \gamma^\mu \phi_i C_{\mu\nu} \bar{\phi}_j \gamma^\nu \phi_j \rangle + \langle \bar{\phi}_i \gamma^\mu \phi_j C_{\mu\nu} \bar{\phi}_j \gamma^\nu \phi_i \rangle$$



## Background field method (Large $N$ )

Translate  $\psi \rightarrow \psi + \phi$  by a field  $\phi \in \mathcal{G}_{0\Lambda}$

$$e^{W^{0\infty}(\eta, \bar{\eta})} = e^{-I^\Lambda(\bar{\phi}, \phi) + \langle \bar{\phi} \eta \rangle + \langle \bar{\eta} \phi \rangle + \tilde{W}^{0\Lambda}(\eta - \delta_{\bar{\phi}} I^\Lambda(\bar{\phi}, \phi), \bar{\eta} + \delta_\phi I^\Lambda(\bar{\phi}, \phi))}$$

$$e^{\tilde{W}^{0\Lambda}(\eta, \bar{\eta})} = \int d\mu_{\bar{\psi}\psi}^{0\Lambda} e^{-\sum_{n \geq 2} L_n^{\Lambda\infty}(\bar{\psi}, \psi) + \langle \bar{\eta} \psi \rangle + \langle \bar{\psi}, \eta \rangle}$$

$$L_n^{\Lambda\infty}(\bar{\psi}, \psi) = \sum_{j+i=n} \frac{(\psi \delta_\phi)^i (\bar{\psi} \delta_{\bar{\phi}})^j}{i! j!} L^{\Lambda\infty}(\bar{\phi}, \phi)$$

$$I^\Lambda(\bar{\phi}, \phi) = \langle \bar{\phi} S_{0\Lambda}^{-1} \phi \rangle + L^{\Lambda\infty}(\bar{\phi}, \phi)$$

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$$\Gamma^{0\infty}(\bar{\phi}, \phi) = I^\Lambda(\bar{\phi}, \phi) + \tilde{\Gamma}^{0\Lambda}(0)$$

$$\Gamma^{0\infty}(\bar{\phi}, \phi) = I^\Lambda(\bar{\phi}, \phi) - \frac{\hbar}{2} \text{Tr} \log [\Gamma''^{0\infty}(\bar{\phi}, \phi) \mathcal{S}_{0\Lambda}] + O(\hbar^2)$$

## Perturbation theory: a valence electron

$$\hat{H} = \alpha^k \partial_k - \frac{\alpha Z}{r} + \beta m - \alpha \frac{\left[ \sum_{i=1}^{N-1} \varphi_i(\vec{x}) \varphi_i^+(\vec{y}) + \rho(\vec{x}, \vec{y}) \right]}{|\vec{x} - \vec{y}|} +$$

$$+ \alpha \int_{\vec{z} \in \mathbb{R}^3} \frac{\text{Tr} \left[ \sum_{i=1}^{N-1} \varphi_i(\vec{z}) \varphi_i^+(\vec{z}) + \rho(\vec{z}, \vec{z}) \right]}{|\vec{x} - \vec{z}|}$$

Using the bilinear formula for the resolvent  $(\hat{H} - \omega) \bar{g}_\omega = \delta^3$

$$\bar{g}_\omega(\vec{x}, \vec{y}) = \int dz \frac{\varphi_z(\vec{x}) \varphi_z^+(\vec{y})}{z_\mp - \omega}, \quad z_\mp = \begin{cases} z - i\epsilon & z \in \sigma_+(\hat{H}), \\ z + i\epsilon & \text{otherwise,} \end{cases} \quad \hat{H} \varphi_z = z \varphi_z$$

where  $\sigma_+(\hat{H})$  is the essential and discrete spectra of unoccupied orbitals.

## Multiscale expansion

Given a generating functional  $L_b^{\Lambda_b \infty}$  and a propagator  $\mathcal{S}_{\Lambda_a \Lambda_b}^b$  we have

$$e^{-L_b^{\Lambda_a \infty}(\Phi)} = \int d\mu_{\Lambda_a \Lambda_b}^b e^{-L_b^{\Lambda_b \infty}(\cdot + \Phi)} = \left( \frac{\det \mathcal{S}_{\Lambda_a \Lambda_b}^b}{\det \mathcal{S}_{\Lambda_a \Lambda_b}^a} \right)^{\frac{1}{2}} \int d\mu_{\Lambda_a \Lambda_b}^a e^{-L_a^{\Lambda_b \infty}(\cdot + \Phi)}$$

$$L_b^{\Lambda_a \infty}(\Phi) = L_a^{\Lambda_a \infty}(\Phi) - \frac{1}{2} \text{Tr} \log(\mathcal{S}_{\Lambda_a \Lambda_b}^{a-1} \mathcal{S}_{\Lambda_a \Lambda_b}^b)$$

We get an expansion with an appropriate propagator at each scale

$$\Gamma^{0\Lambda}(0) = -\frac{\hbar}{2} \lim_{K \rightarrow \infty} \sum_{k=1}^K \text{Tr} \log \left[ (\mathcal{S}_{\Lambda_k \Lambda_{k+1}}^k)^{-1} \mathcal{S}_{\Lambda_k \Lambda_{k+1}}^{k+1} \right] + O(\hbar^2)$$

## Hypothesis 1

Only corrections to the mass and the nuclear charge,  $\Gamma''(0) = -\Omega^\lambda$

$$\beta(\Omega^\lambda \varphi)_x = \beta \int_{\vec{z} \in \mathbb{R}^3} \mu_{\vec{z}}^\lambda \varphi_{\vec{x}-\vec{z}} + \alpha \int_{\vec{z} \in \mathbb{R}^3} \frac{\rho_{\vec{z}}^\lambda}{|\vec{x}-\vec{z}|} \varphi_{\vec{x}}$$

$\mu^\lambda(r)$ ,  $\rho^\lambda(r)$  are spherically symmetric functions

$$\hat{\rho}^\lambda(p) = \int_0^\infty r^2 \rho^\lambda(r) e^{-prm}, \quad \hat{\mu}^\lambda(p) = \int_0^\infty r^2 \mu^\lambda(r) e^{-prm}$$

At  $\lambda = \Lambda$  QED at 1-loop. A low energy decomposition,  $p < 1$ :

$$\hat{\rho}^\lambda(p) = \rho_1^\lambda p + \rho_2^\lambda p^2 + \rho_3^\lambda p^3 + \rho_4^\lambda p^4 \log p + O(p^5)$$

$$\hat{\mu}^\lambda(p) = \mu_0^\lambda + \mu_1^\lambda p + \mu_2^\lambda p^2 + \mu_3^\lambda p^3 + \mu_4^\lambda p^4 \log p + O(p^5)$$

## Hypothesis 2

For some  $M \geq N$  we have a set of solutions  $\{\varphi_i \in \mathbf{H}^1\}_{i=1}^M$

$$\hat{H}\varphi_n = z_n\varphi_n, \quad \hat{H} = \alpha^k \partial_k - \frac{Z\alpha}{r} + \beta(m + V - \Sigma)$$

$0 < \lambda \leq z_M$ :

$$\Gamma^{\lambda\Lambda}(\bar{\psi}, \psi) = \langle \bar{\psi} \Omega^\lambda \psi \rangle - \frac{\langle \bar{\psi} Q \bar{\psi} \rangle + \langle \psi \bar{Q} \psi \rangle}{2} + \sum_{n>2} L_n^{\lambda\Lambda}$$

$$R_1^{\lambda\Lambda} = -\sigma_{\lambda\Lambda} \sum_{n=1}^M \varphi_n \varphi_n^+$$

$z_M < \lambda \leq \Lambda$ :

$$\Gamma^{\lambda\Lambda}(\bar{\psi}, \psi) = \langle \bar{\psi} \Omega^\lambda \psi \rangle + L_4^{\lambda\Lambda}$$

where  $R_1^{\lambda\Lambda}$  corresponds to the Green function with  $Z = 1$ , R.A. Swainson and G. W. F. Drake.

## Non-perturbative RG equations

One consider the renormalization group equation for two point function.  
At 1-loop order we have

$$\dot{\Gamma}^{\Phi_1\Phi_2} = \frac{\hbar}{2} \left[ \langle \dot{\sigma}_{\lambda\Lambda} \Gamma''^{\Phi_1\Phi_2} \mathcal{S}_{\lambda\Lambda} \rangle + \langle \dot{\sigma}_{\lambda\Lambda} \Gamma''^{\Phi_1} \mathcal{S}_{\lambda\Lambda} \Gamma''^{\Phi_2} \mathcal{S}_{\lambda\Lambda} \rangle - \langle \Phi_1 \leftrightarrow \Phi_2 \rangle \right]$$

By smearing the both sides with solutions of hydrogen atom of appropriate quantum numbers one obtains the necessary number of ODEs for the coefficients  $\rho_i^\lambda, \mu_i^\lambda$ .