On the Theory of the Electron and Positive

W. H. FURRY* AND J. R. OPPENHEIMER, *University of California, Berkeley, California*

(Received December 1, 1933)

In this paper we develop Dirac's suggestions for the interpretation of his theory of the electron to give a consistent theory of electrons and positives. In Section 1, we discuss the physical interpretation of the theory, the limits which it imposes on the spatiotemporal description of a system and in particular on the localizability of the electron. In Section 2, we set up the corresponding formalism, introducing wave functions to describe the state of the electrons and positives in the system, and constructing operators to represent the energy, charge and current density, etc. It is shown that the theory is Lorentz invariant, and has just that invariance under contact transformations which the physical interpretation requires. The electromagnetic interaction of the electrons and positives is formulated, and certain ambiguities which arise here are discussed. In Section 3, it is shown that in all problems to which the Dirac equation is directly applicable it gives the correct energy levels for an electron, and the correct radiative and collision transition probabilities. In these problems the wave functions are constructed from the solutions of the Dirac equation. In Section 4, we discuss certain problems which have no analogue in the original Dirac theory of the electron, show that a certain part of the energy of an electromagnetic field in general resides in the electrons and positives, and consider the extent to which, in the present state of theory, this can be detected by experiment. For two charges within a Compton wave-length of each other the law of force is not quite Coulomb's law. The deviations though small should in principle be detectable when protons are scattered in hydrogen.

1.

WITH the discovery of the positive electron, one of the most curious and radical predictions of Dirac's theory of the electron has received experimental support. It appears that the positive electron has the properties of the anti-electron of the Dirac theory, and that in some cases, notably in the absorption and internal conversion of gamma-rays, the circumstances of the production of pairs of electrons and positives are just those to be expected from the theory. We want to develop the formalism of the theory in its present form as clearly and consistently as possible, and to show how the formalism is to be interpreted physically, where it justifies earlier methods of calculation, where it leads to new predictions, and where these predictions fail. We shall see that the present theory is limited in very much the same way as the quantum theory of wave fields: here, too, it is possible to develop a consistent and relativistic formalism which within wide limits corresponds to the possibilities of physical observation and gives correct results; but here, too, the legitimate application of the theory is limited to lengths which are large compared to the classical electron radius e^2/mc^2; and when not so limited, leads at once to grave contradictions with experiment.

The formal changes which are required in the theory are simple, and correspond closely to Dirac's most recent suggestion[1] for interpreting the negative kinetic energy states which are falsely predicted by his theory of the electron in its original form. These formal changes may most easily be formulated with the help of the distinction made by Schroedinger,[2] in his attempt to resolve the difficulties with the negative energy states, between odd and even operators, and show how this distinction is properly to be applied to give a consistent and Lorentz invariant theory. This distinction between even and odd operators is essential to the physical interpretation of the theory: for only those dynamical variables of a system which correspond to even operators may be determined without rendering the number of particles—electrons and positives—in the system indeterminate.

* National Research Fellow.

[1] P. A. M. Dirac, Proc. Roy. Soc. **A133**, 60 (1931).

[2] E. Schroedinger, Berl. Ber., p. 63, 1931. For a review of these earlier theories of the Dirac electron, see W. Pauli, Handb. d. Physik **24**, XXIV, 242–247 (1933).

According to Dirac's theory, the states of an electron subject to no forces are given by the solutions of the wave equation

$$i\hbar\dot{\psi}=H_0\psi; \qquad H_0=-\beta mc^2-c(\boldsymbol{\alpha}\cdot\mathbf{p}). \quad (1.1)$$

Any solution of (1.1) may be built up out of solutions of two sorts: those of positive kinetic energy, for which

$$i\hbar\dot{\psi}=H_0\psi=E\psi, \quad \text{with} \quad E\geq mc^2 \quad (1.2)$$

and those of negative kinetic energy:

$$i\hbar\dot{\psi}=H_0\psi=E\psi, \quad \text{with} \quad E\leq -mc^2. \quad (1.3)$$

Any state which is a superposition of states of type (1.2) may be called positive, any of type (1.3) alone negative. Now with the help of these solutions, we can assign a matrix to represent any function of the elementary dynamical variables of the electron: p, x, α. When this matrix reduces to two submatrices for positive and negative states respectively and has no elements in which one index refers to positive, the other to negative states, the corresponding variable is said to give us an even operator. The momenta of an electron are even; the coordinates are not.

Variables which correspond to operators which are not even may still, of course, in principle be determined; important examples of such variables are the total energy of a system in an external field, and the charge and current density of a system; but this cannot be done without introducing an indeterminacy in the number of electrons and positives present. This circumstance which corresponds to a complementarity between the description of a system in terms of a definite number of permanent particles on the one hand, and a spatiotemporal specification of these particles on the other, is decisive in limiting the applicability of the nonrelativistic transformation theory of the quantum mechanics. For it is not, in general, possible to specify at the same time the number of particles in a system and the values of arbitrary dynamical variables of these particles. In particular, this means that the unambiguous determination of the presence of a positive in a system demands that under the circumstances of the experiment one should be able to abstract entirely from the creation and destruction of pairs. This must particularly be borne in mind in putting such questions as "Are there any positives present near a nucleus?"; for such a question can be given an unambiguous meaning only by reference to such an experimental arrangement for answering it, that one may abstract from the nuclear field of the atom which produces and destroys pairs.

From the formal point of view this situation implies serious limitations on the use of wave functions defined in a configuration space: wave functions whose arguments are the characteristic values of the dynamical variables of the particles of the system. For, quite apart from the limitation of the symmetry character of the wave functions imposed by the exclusion principle, we meet here for the first time the requirement that such wave functions be defined as functions only of even variables, since no physical interpretation whatever can be given to the probability that a particle shall have a determinate value for a variable which is not wholly even. The canonical transformations of the configuration space, which in nonrelativistic theory are unrestricted, are here limited to even transformations. In general, too, a wave function defined in such an even configuration space will not be adequate to describe the state of the system, since the number of particles in the system, and thus the dimensions of the configuration space, will not in general be determinate: here, as in the description of the electromagnetic field, we shall need, in general, a series of wave functions, defined in spaces of different numbers of dimensions and corresponding to the probability of finding different numbers of particles in the system. This circumstance means that, again as in the case of the electromagnetic field, the formal aspects of the theory are best studied by abandoning the wave functions defined in configuration space and by using the method of quantized waves in actual space; for specific calculations it is more convenient to revert to the configuration space functions; and it is, of course, possible to establish a correspondence between the functions introduced by the two methods.

Before we develop this formalism, we may illustrate these considerations in the important case of the position determinations of the electron. It is important to note that what we say here has to do only with the critical length $\hbar/mc$, and throws no light at all on the breakdown of the theory for lengths of the order e^2/mc^2. For what we have here to consider, the charge on the electron could have an arbitrarily small value;

and these considerations, like all those based on the present relativistic quantum mechanics, must rather be regarded as attaining asymptotic validity when the ratio of the two critical lengths, which is the fine structure constant, α, goes to zero.

It has often been observed[3] that by the gamma-ray microscope one could not, in general, determine the position of an electron with arbitrary precision. The limits on this precision are determined by the, in general, finite wave-length of light scattered through a large angle by the electron. For an electron at rest this wave-length is of the order of the Compton wave-length λ_0; for an electron whose kinetic energy before the scattering is E, it is possible to choose an arrangement such that the wave-length of the light scattered through a right angle is $\sim hc/E$. The finiteness of these wave-lengths limits, of course, the precision with which, from a given observation in the microscope, one can infer the position of the scattering electron. This limitation inheres in all attempts to localize the electron by scattering experiments, whatever be the nature of the scattered radiation.

The Dirac theory of the electron, on the other hand, starts with the postulation of a probability density $W(x)$ that the electron be found near the point x, and thus guarantees the observability of the position of the electron. But it does this only at the expense of admitting the existence of states of negative kinetic energy. For the conclusion that hard light scattered by an electron is necessarily softened by the scattering depends essentially upon the fact that the kinetic energy of the electron before the scattering is taken positive. Because of the nonexistence in fact of electrons of negative kinetic energy, the postulation of the complete localizability of the electron and the existence of the probability density $W(x)$ appears unjustifiable.

With the charge density the situation is radically different. On the Dirac theory, it is true, this charge density is merely proportional to $W(x)$:

$$\rho(x) = eW(x). \tag{1.4}$$

[3] See for instance L. Landau and R. Peierls, Zeits. f. Physik **69**, 56 (1931).

But for the determination of ρ other experimental procedures are available. For the quantum theory of the electromagnetic field and the careful consideration recently given by Bohr to the possibilities of observation which it implies show that, at least so far as we may abstract from the atomic nature of the measuring instruments, the electric field may be mapped out with any precision we want; its divergence gives the charge density ρ, which may by this method be in principle determined. In any theory in which the atomic nature of measuring instruments is neglected, this observability of charge density must persist. Since we have seen what grave difficulties inhere in relativistic theory in the definition of the particle density, we must be prepared to abandon the simple definition of ρ given by (1.4). As a matter of fact we shall see that in the present theory the vanishing of the mean charge density in a region of space does not necessarily mean that there are no particles present in the region; it may mean only that there are equal numbers of electrons and positives present and is no guarantee that the region is empty. For in the present theory, the charge density is not an even operator and can therefore be determined only at the expense of leaving indeterminate the number of particles present in the system.

As far as the localizability of the electron itself is concerned, the conclusions reached by the consideration of the gamma-ray microscope are fully confirmed by theory. For this it is essential to observe that the coordinate of an electron is not an even operator; this means that in the Dirac theory it is not possible at once to specify the coordinate of a particle and to be sure that its kinetic energy is positive. Now in the determination of the position of an electron, we cannot admit an indeterminacy in the number of particles present since we must be sure that the light was scattered into the microscope by the electron which we wish to observe. (We are considering for simplicity the case where only a single electron is initially present.) We cannot, therefore, devise an experiment for measuring precisely the coordinate of the electron. What we have to consider is how precisely we can infer the position of an electron from an experiment in which an even operator is determined. Thus we may determine the "even part" of the coordinate which is

given as an operator by

$$g(x)=x-(\hbar c/2i)[\alpha_x H_0^{-1}+cp_x H_0^{-2}]. \quad (1.5)$$

Inferences as to the value of the coordinate depend upon the properties of the transformation functions $g\rightarrow x$. More precisely, these four-component functions are the transformation functions from g, σ the component of spin in a fixed direction and the sign of the kinetic energy to the coordinates.[4] They are given by

$$(x, i\,|\,g, \sigma, +)=2^{-\frac{1}{2}}\int_{-\infty}^{\infty}dp\,\exp\,[i(x-g)p/\hbar]$$
$$\times[(m^2c^2+p^2)^{\frac{1}{2}}\{mc+(m^2c^2+p^2)^{\frac{1}{2}}\}]^{-\frac{1}{2}}a_i^{\sigma}, \quad (1.6)$$

where

$$a_1^1=-p, \qquad a_2^2=p,$$
$$a_3^1=mc+(m^2c^2+p^2)^{\frac{1}{2}}, \qquad a_4^2=mc+(m^2c^2+p^2)^{\frac{1}{2}},$$
$$a_2^1=a_4^1=0, \qquad a_1^2=a_3^2=0,$$

where these functions are taken for positive kinetic energy, and $\sigma=1$ corresponds to spin parallel, $\sigma=2$ to spin antiparallel, to the x axis; a set of functions for negative kinetic energy, which are relevant only in discussing the localization of the positive, may be obtained from (1.6) by replacing $(m^2c^2+p^2)^{\frac{1}{2}}$ throughout by $-(m^2c^2+p^2)^{\frac{1}{2}}$.

These functions (1.6) form a complete set for the expansion of any positive function; and the mean dispersion, defined in the usual way as $\overline{x^2}-\bar{x}^2$ vanishes for all these packets.[5] This corresponds to the circumstance that it is possible to prepare an electron in such a way that its position may be known without uncertainty. But from the results of the experiment, namely the determination of g, no unambiguous and precise inference about the position of the electron before the measurement may be drawn. For the probability of finding a value g is determined by

$$P(g)\Delta g=\Delta g$$
$$\times\sum_{\sigma=1}^{2}\left|\int_{-\infty}^{\infty}dx\sum_{i=1}^{4}(g, \sigma, +\,|\,x, i)\psi_i(x)\right|^2 \quad (1.7)$$

and this is not in general given by

$$W(g)\Delta g=\Delta g\cdot\sum_{i=1}^{4}|\,\psi_i(g)\,|^2 \quad (1.8)$$

but depends upon the value of ψ in a region of the order λ_0 about g. When nothing is known about the state of the electron, ψ, before the measurement, the value of the coordinate x is fixed by the determination of g only within the limit of the Compton wave-length. When it is known that the kinetic energy of the electron before the observation is certainly as great as E, then the probability of observing a given value for g depends on the magnitude of the wave function ψ only throughout the region $g\pm hc/E$; and the position of the electron before the experiment can be inferred from the results of the experiment with a correspondingly greater precision. One may say therefore that, whereas it is possible by experiment to localize an electron, in the sense that the position of the electron after the experiment may be determined as precisely as one wishes, it is nevertheless not possible to determine the position of an electron in an arbitrary state with a precision greater than λ_0. It will be seen that this corresponds exactly to the possibilities offered by the gamma-ray microscope where the initial determination of position is necessarily unprecise by about hc/E, but where a second determination of position, after the electron is known to have scattered a hard gamma-ray, may in principle be made precise.

We have in this discussion assumed, in accordance with the physical considerations adduced and in conformity with the theory which we are here to develop, that the states of an electron in a system where it is known that no positives or pairs are produced correspond to purely positive functions; in the same way we shall see that the states of the isolated positive correspond to purely negative states; and the limits here derived for the localizability of an electron hold unchanged for a positive. In situations in which, as in the determination of the charge density, we cannot be sure of the number of charges present in our system, no unambiguous position measurements of the charges are possible.

Because of these considerations it is not in general possible, in experimental situations where electron pairs are produced, to localize the region

[4] We here give the formulae for the x-coordinate, and take $p_y=p_z=0$: analogous formulae can be given which correspond to determining $g(x)$, $g(y)$ and $g(z)$ at once.

[5] This is apparently in disagreement with the argument of Landau and Peierls, reference 3.

in which they are produced with a precision much greater than the Compton wave-length. In this sense the production of pairs by the absorption of gamma-rays and their internal conversion must be regarded as an extranuclear phenomenon; as far as present theory is concerned, the fields within the nuclei have singularly little effect upon the probability of these processes.

We must turn now to the development of the formal theory. In the next section we shall introduce the necessary methods, set up the wave equations which the wave functions of the theory must satisfy, and give the necessary proofs of invariance. In Section 3 we shall show how these equations may be solved and how their solutions are connected with those of the original Dirac theory, and justify the use of this theory in most of the cases to which it has been applied. In Section 4 we must discuss certain new effects introduced in the present theory and define those limitations on the applicability of the theory which are introduced by the here postulated and not understood corpuscular nature of the elementary charges.

2.

We must now find what wave functions describe the states and what operators correspond to the dynamical variables, in particular the energy and charge and current density, of a system of electrons and positives. In the non-relativistic quantum mechanics, a system of N indistinguishable particles which satisfy the exclusion principle may be described by an anti-symmetric wave function in the $3N$-dimensional configuration space of the particles:

$$\psi(x_1 y_1 z_1 \cdots x_N y_N z_N). \tag{2.1}$$

We can find the wave function corresponding to this state as a function of any other complete set of commuting variables $\xi_1, \xi_2 \cdots \xi_{3N}$ with the help of the transformation functions

$$(\xi_1 \cdots \xi_{3N} \,|\, x_1 \cdots z_N):$$

$$\psi(\xi_1 \cdots \xi_{3N}) = \int \cdots \int dx_1 \cdots dz_N (\xi_1 \cdots \xi_{3N} \,|\, x_1 \cdots z_N) \psi(x_1 \cdots z_N). \tag{2.2}$$

In particular we can specify the configuration of the jth electron by the three commuting variables ξ_j, η_j, ζ_j instead of x_j, y_j, z_j, and use the wave functions

$$\psi(\xi_1 \cdots \zeta_N) = \int \cdots \int dx_1 \cdots dz_N \prod_{j=1}^{N} (\xi_j \eta_j \zeta_j \,|\, x_j y_j z_j) \psi(x_1 \cdots z_N) = \psi(r_1 \cdots r_N) \tag{2.3}$$

for the system. It will in no way impair the generality of the argument to suppose the sets of characteristic values of ξ, η, ζ enumerable; we can then order these sets of characteristic values in an arbitrary but fixed order; and for any set we can use the symbol r. We can then fix the sign of the antisymmetric function (2.3) by letting its arguments be ordered. If we introduce[6] new variables N_r to describe the system, one for each characteristic value of r, and let $N_r = 1$ when there is a particle present in the state r, $N_r = 0$ when there is no particle present in this state, then we can describe the state of the system by a wave function of the N_r's, defined by

$$\psi(N_r) = 0 \quad \text{when} \quad \sum_r N_r \neq N$$

$$\psi(0 \cdots 1_{r_1} \cdots 0 \cdots 1_{r_i} \cdots 1_{r_N} \cdots) = (N!)^{\frac{1}{2}} \psi(r_1 \cdots r_N). \tag{2.4}$$

Any dynamical variable of the system which is symmetric in the particles corresponds to an operator on (2.4). To a variable which, as an operator on (2.3), is given by

$$\sum_{j=1}^{N} \Omega_{r_j' r_j} \tag{2.5}$$

there corresponds the operator

$$\sum_{r,\, r'} \Omega_{r' r} a_{r'}^{+} a_r. \tag{2.6}$$

The a_r, a_r^{+} here introduced satisfy

[6] P. Jordan and E. Wigner, Zeits. f. Physik **47**, 631 (1928).

$$a_r a_s + a_s a_r = 0; \quad a_r^+ a_s^+ + a_s^+ a_r^+ = 0;$$
$$a_r^+ a_s + a_s a_r^+ = \delta_{rs}. \quad (2.7)$$

They may be expressed in terms of the N_r's and operators Γ_r, v_r:

$$\Gamma_r F(N_1 \cdots N_{r-1}, N_r, N_{r+1}, \cdots) = F(N_1 \cdots N_{r-1}, 1-N_r, N_{r+1} \cdots)\Gamma_r;$$
$$v_r F(N) = (-)^{\Sigma_{s=1}^{r} N_s} F(N) \quad (2.8)$$

by

$$a_r = v_r \Gamma_r N_r; \quad a_r^+ = N_r \Gamma_r v_r. \quad (2.9)$$

Thus

$$N_r = a_r^+ a_r; \quad 1-N_r = a_r a_r^+;$$
$$N_r^2 = N_r; \quad \Gamma_r^2 = 1; \quad v_r^2 = 1. \quad (2.10)$$

Similarly, to an operator on (2.3) of the form

$$\sum_{i,j} \Omega_{r_i' r_j'; r_i r_j} \quad (2.11)$$

there corresponds the operator

$$\sum_{r', s', r, s} \Omega_{r's'; rs} a_{r'}^+ a_r a_{s'}^+ a_s \quad (2.12)$$

and to

$$\sum_{i \neq j} \Omega_{r_i' r_j'; r_i r_j} \quad (2.13)$$

the operator

$$\sum_{r', s', r, s} \Omega_{r's'; rs} a_{r'}^+ a_{s'}^+ a_r a_s. \quad (2.14)$$

It is in terms of this formalism that the relativistic theory of electrons and positives may most easily be developed. The formalism as it stands is immediately applicable to the Dirac theory of the electron in its original form and describes a system of N electrons whose wave functions are given by the Dirac wave equation, and in which the electrons may have positive or negative kinetic energies. (To describe the state of an electron, r must be supposed to include a variable which determines the orientation of the spin of the electron.) But a simple formal change, which corresponds exactly to regarding the emptiness and not the fullness of a state of negative kinetic energy as equivalent to the presence of a particle, gives us at once the theory of electrons and positives. To make this change it is necessary, in accordance with the physical considerations of Section 1, to restrict the r's in a critical way. From now on we shall consider only states r, such that a given state is either wholly positive or wholly negative; the states r which are positive shall be indexed by Latin letters; the negative by Greek; and a state which may be either positive or negative will be indexed by a Latin letter in parentheses: (r). By this restriction the unlimited applicability of the quantum-mechanical transformation theory is destroyed.

For what follows it is helpful to generalize our definition of positive and negative states. A positive state is one which is a superposition of states of a free particle of positive kinetic energy. And the states of a free particle are determined by a wave equation which is in general

$$(c\pi_0 - T)\psi = 0; \quad T = -c(\boldsymbol{\alpha} \cdot \boldsymbol{\pi}) - \beta mc^2. \quad (2.15)$$

Here

$$\pi_0 = (ih/c)(\partial/\partial t) - (e/c)V;$$
$$\boldsymbol{\pi} = -ih \operatorname{grad} - (e/c)\mathbf{A} \quad (2.16)$$

and, for no field

$$V = -(1/c)(\partial/\partial t)\lambda(x, y, z, t);$$
$$\mathbf{A} = \operatorname{grad} \lambda(x, y, z, t).$$

The solutions of (2.15) are given in terms of the solution ψ_0 of (2.15) with $\lambda = 0$ by the transformation

$$\psi = \psi_0 \exp(ie\lambda/\hbar c).$$

We have of course to consider solutions of (2.15) which correspond to definite values of the kinetic energy, which is represented by the operator $c\pi_0 = T$. It will be observed that under the transformation

$$V' = V - \frac{1}{c}\frac{\partial \lambda}{\partial t}; \quad \mathbf{A}' = \mathbf{A} + \operatorname{grad} \lambda;$$
$$\psi' = \psi \exp(ie\lambda/\hbar c) \quad (2.17)$$

the kinetic energy is invariant; under it positive states go into positive, negative into negative states.

Let us consider an operator of the form (2.5) for which $\Omega_{(r)(s)}$ is a diagonal matrix: to it will correspond the operator

$$\sum_r \Omega_{rr} N_r + \sum_\rho \Omega_{\rho\rho} N_\rho \quad (2.18)$$

or, if we write $M_\rho = 1 - N_\rho$,

$$\sum_r \Omega_{rr} N_r - \sum_\rho \Omega_{\rho\rho} M_\rho + \sum_\rho \Omega_{\rho\rho}. \quad (2.19)$$

For a system containing N electrons—i.e., for which N of the N_r for the occupied states are 1, and the others vanish—and containing M positives, in states corresponding to $\rho_1 \cdots \rho_M$, we should expect for the operator (2.6) just

$$\sum_r \Omega_{rr}N_r+\sum_\rho(-\Omega_{\rho\rho})M_\rho. \qquad (2.20)$$

We are thus led to consider, in connection with an arbitrary operator Ω, the operator

$$\hat{\omega}=\sum_{(r)(s)}\Omega_{(r)(s)}\omega_{(r)(s)} \qquad (2.21)$$

with

$$\omega_{r(s)}=a_r{}^+a_{(s)};\quad \omega_{(r)s}=a_{(r)}{}^+a_s;$$
$$\omega_{\rho\sigma}=a_\rho{}^+a_\sigma-\delta_{\rho\sigma}=-a_\sigma a_\rho{}^+. \qquad (2.22)$$

This assignment suffices for a consistent description of a system of noninteracting electrons and positives. To the question of the treatment of the interaction of these particles, which cannot be wholly clarified, and to which no complete and unambiguous answer can be given, we shall return only later.

The energy of a system of noninteracting particles, which may however be subject to arbitrary external forces, is of the form (2.5), and corresponds then to the operator

$$\mathcal{H}=\sum_{(r)(s)}H_{(r)(s)}\omega_{(r)(s)}. \qquad (2.23)$$

In particular the kinetic energy $\mathfrak{T}$ of the system is

$$\sum_r N_rT_{rr}-\sum_\rho M_\rho T_{\rho\rho} \qquad (2.24)$$

and is necessarily positive, the sum of the positive kinetic energies of the electrons and the positive kinetic energies $-T_{\rho\rho}$ of the positives. The difference between the total number of positives and electrons in the system, which gives the total charge,

$$-e\nu=e(M-N);\quad M=\sum_\rho M_\rho;\quad N=\sum_r N_r$$

commutes with $\mathcal{H}$, and is constant, as it should be: to any system there corresponds a fixed value of ν. But both M and N change with time in general:

$$-i\hbar\dot{N}\equiv-i\hbar\dot{M}\equiv\sum_{r\rho}(H_{\rho r}a_\rho{}^+a_r-H_{r\rho}a_r{}^+a_\rho). \qquad (2.25)$$

Unless the energy corresponds to an even operator the number of particles in the system $N+M$ will not be constant. The rate of production of a pair r, ρ is measured by $|H_{r\rho}|^2$ and that of its annihilation by $|H_{\rho r}|^2$.

The modification by which (2.21) was obtained from (2.6), namely the subtraction of

$$\sum\nolimits_-(\Omega)=\sum_\rho\Omega_{\rho\rho} \qquad (2.26)$$

has destroyed, as we have seen, and as the physics requires, the invariance of the theory under arbitrary contact transformations. If we restrict ourselves however to the even transformations, $((r)|(s))$, for which $((r)|(s))$ is an even matrix, then $\sum_-(\Omega)$, which is the spur of the matrix $\Omega_{(r)(s)}$ in an invariant subspace, will remain invariant. To any wave function of the N_r and M_ρ there will correspond a wave function of N_s and M_σ for any s, σ which are obtained from r, ρ by an even transformation; and under such transformation the theory is invariant.

The Lorentz invariance of the theory follows from this. For under Lorentz transformation positive kinetic energies go over into positive, negative into negative; the Lorentz transformation is even. Thus

$$\sum\nolimits_-(\Omega)\quad\text{and}\quad\hat{\omega}=\sum_{(r)(s)}\Omega_{(r)(s)}a_{(r)}{}^+a_{(s)}-\sum\nolimits_-(\Omega)$$

will transform under a Lorentz transformation just as Ω does; if Ω is a scalar, then

$$\sum\nolimits_-(\Omega)\quad\text{and}\quad\hat{\omega}$$

will be scalars; if the Ω^μ form a four vector, so will the

$$\sum\nolimits_-(\Omega^\mu)\text{ and the }\hat{\omega}^\mu.$$

In the present formulation the subtraction of the spur (2.26) destroys the gauge invariance of the theory, in that it is not possible to associate in an unambiguous way with a given choice of gauge to represent a given field a unique gauge in (2.16). This circumstance means that the division of states into positive and negative here formulated is still provisional, and will have to be modified in the presence of an electromagnetic field. As the arguments of the next section will show, the non-gauge-invariance of the theory does not affect the predictions which the theory makes about the reactions of the system to a given external field—the energy differences of the states of the system in the field, the transition probabilities, and the probabilities of production of observable pairs. On the other hand, the predictions of the theory about the reaction of the system on the electromagnetic field are gauge-dependent.

Again with neglect of the interaction of the particles, we may write the wave equation for the functions $\psi(N_r; M_\rho)$. If there is an electromagnetic field described by the potentials V, A then the Hamiltonian is given by

$$H=eV+i\hbar c(\mathfrak{a}, \text{grad})+e(\mathfrak{a}\cdot A)-\beta mc^2. \quad (2.27)$$

To this there corresponds the matrix $H_{(r)(s)}$ and to this the operator

$$\underset{\sim}{H}=\sum_{(r)(s)} H_{(r)(s)}\omega_{(r)(s)}. \quad (2.28)$$

The wave equation is then

$$i\hbar(\partial/\partial t)\psi(N_r; M_\rho)=\underset{\sim}{H}\psi(N_r; M_\rho). \quad (2.29)$$

We can readily write the wave equation in the configuration space of the particles which corresponds to this. We define the wave functions in configuration space, in analogy to (2.4), by[7]

$$\psi(N_r; M_\rho)=0, \qquad N-M\neq\nu$$
$$\psi(0\cdots 1_{r_1}0\cdots 1_{r_2}\cdots 1_{r_N}\cdots; 0\cdots 1_{\rho_1}\cdots 0\cdots 1_{\rho_M}\cdots)=(N!M!)^{\frac{1}{2}}(-)^{\sum_j^M [\rho_j]}\psi_{NM}(r_1\cdots r_N; \rho_1\cdots\rho_M), \qquad N-M=\nu \quad (2.30)$$

with the arguments of the ψ_{NM} ordered. We may now define the operator $\hat{\omega}$ corresponding to (2.5) and (2.21) as an operator in configuration space. This operator transforms any function ψ into $\hat{\omega}\psi$, according to

$$(\hat{\omega}\psi)_{N,M}(r;\rho)=\sum_{r'}\sum_{i}^{N}\Omega_{r_i r'}\psi_{N,M}(\cdots r_{i-1}r'r_{i+1}\cdots;\rho)-\sum_{\rho'}\sum_{j}^{M}\Omega_{\rho'\rho_j}\psi_{N,M}(r;\cdots\rho_{j-1}\rho'\rho_{j+1}\cdots)$$
$$-(NM)^{-\frac{1}{2}}\sum_{i}^{N}\sum_{j}^{M}(-)^{N-i+j-1}\Omega_{r_i\rho_j}\psi_{N-1,M-1}(\cdots r_{i-1}r_{i+1}\cdots;\cdots\rho_{j-1}\rho_{j+1}\cdots)$$
$$-(N+1)^{\frac{1}{2}}(M+1)^{\frac{1}{2}}\sum_{r'}\sum_{\rho'}\Omega_{\rho' r'}\psi_{N+1,M+1}(r_1\cdots r_N r';\rho'\rho_1\cdots\rho_M). \quad (2.31)$$

Then the wave equation is

$$i\hbar(\partial/\partial t)\psi_{N,M}(r;\tau)=(\underset{\sim}{H}\psi)_{N,M}(r;\rho) \quad (2.32)$$

or, in detail, for $N-M=\nu=0$:

$$i\hbar(\partial/\partial t)\psi_{00}=-\sum_{r'}\sum_{\rho'}H_{\rho' r'}\psi_{11}(r';\rho'), \quad (2.33)$$

$$i\hbar(\partial/\partial t)\psi_{11}(r_1;\rho_1)=\sum_{r'}H_{r_1 r'}\psi_{11}(r';\rho_1)-\sum_{\rho'}H_{\rho'\rho_1}\psi_{11}(r_1;\rho')-H_{r_1\rho_1}\psi_{00}-2\sum_{r'}\sum_{\rho'}H_{\rho' r'}\psi_{22}(r_1 r';\rho'\rho_1);$$

. . . . . .

for $\nu=1$:

$$i\hbar(\partial/\partial t)\psi_{10}(r_1)=\sum_{r'}H_{r_1 r'}\psi_{10}(r')-2^{\frac{1}{2}}\sum_{r'}\sum_{\rho'}H_{\rho' r'}\psi_{21}(r_1 r';\rho'),$$
$$i\hbar(\partial/\partial t)\psi_{21}(r_1 r_2;\rho_1)=\sum_{r'}H_{r_1 r'}\psi_{21}(r' r_2;\rho_1)+\sum_{r'}H_{r_2 r'}\psi_{21}(r_1 r';\rho_1)$$
$$-\sum_{\rho'}H_{\rho'\rho_1}\psi_{21}(r_1 r_2;\rho')-2^{-\frac{1}{2}}H_{r_2\rho_1}\psi_{10}(r_1)+2^{-\frac{1}{2}}H_{r_1\rho_1}\psi_{10}(r_2)-6^{\frac{1}{2}}\sum_{r'}\sum_{\rho'}H_{\rho' r'}\psi_{32}(r_1 r_2 r';\rho'\rho_1); \quad (2.34)$$

. . . . . .

and for $\nu=-1$:

$$i\hbar(\partial/\partial t)\psi_{01}(\rho_1)=-\sum_{\rho'}H_{\rho'\rho_1}\psi_{01}(\rho')-2^{\frac{1}{2}}\sum_{r'}\sum_{\rho'}H_{\rho' r'}\psi_{12}(r';\rho'\rho_1),$$
$$i\hbar(\partial/\partial t)\psi_{12}(r_1;\rho_1\rho_2)=\sum_{r'}H_{r_1 r'}\psi_{12}(r';\rho_1\rho_2)-\sum_{\rho'}H_{\rho'\rho_1}\psi_{12}(r_1;\rho'\rho_2)$$
$$-\sum_{\rho'}H_{\rho'\rho_2}\psi_{12}(r_1;\rho_1\rho')-2^{-\frac{1}{2}}H_{r_1\rho_1}\psi_{01}(\rho_2)+2^{-\frac{1}{2}}H_{r_1\rho_2}\psi_{01}(\rho_1)-6^{\frac{1}{2}}\sum_{r'}\sum_{\rho'}H_{\rho' r'}\psi_{32}(r_1 r';\rho'\rho_1\rho_2). \quad (2.35)$$

. . . . . .

[7] Here $[\rho_j]$ is the ordinal number assigned to the state ρ_j in ordering the negative states. The factor $(-)^{\sum_j^M [\rho_j]}$ is inserted to save writing.

The rule for the transition from (2.6) to (2.21) cannot be unambiguously extended to operators of the form (2.14); and for the consideration of the electromagnetic interaction of the particles of the system we must revert to electrodynamical theory. According to (2.21), to the charge density

$$\sum_i \rho_{(r)_i'(r)_i}; \quad \rho_{(r)'(r)} = e\bar{\psi}_{(r)'}\psi_{(r)} \tag{2.36}$$

and current density

$$\sum_i \mathbf{j}_{(r)_i'(r)_i}; \quad j_{(r)'(r)} = e\bar{\psi}_{(r)'}\boldsymbol{\alpha}\psi_{(r)} \tag{2.37}$$

must be assigned the operators

$$\sum_{(r)(s)} \rho_{(r)(s)}\omega_{(r)(s)} \quad \text{and} \quad \sum_{(r)(s)} \mathbf{j}_{(r)(s)}\omega_{(r)(s)}. \tag{2.38}$$

The form a four vector $\check{S}^\mu$.

The electrostatic interaction of the system with itself, i.e., the Coulomb forces exerted by the particles on themselves and on each other, may then, with neglect of retardation, be obtained from the matrix

$$C_{(r)(s);\,(l)(m)} = \int\int d\tau d\tau' \frac{\rho_{(r)(s)}(x)\rho_{(l)(m)}(x')}{|r-r'|} \tag{2.39}$$

and give an operator

$$\hat{C} = \sum_{(r),\,(s),\,(l),\,(m)} C_{(r)(s);\,(l)(m)}\omega_{(r)(s)}\omega_{(l)(m)}. \tag{2.40}$$

This includes of course in its present form the proper electrostatic energy of the particles.

When the magnetic forces and the retardation of the forces between the charges must be taken strictly into account, there can be, of course, no operator in the configuration space of the particles to represent the corresponding energy; rather one must use the expressions (2.38) for the charge and current of the system, and from these compute the interaction energy between the charges and the electromagnetic field

$$H_I = \int d\tau[\rho V + (\mathbf{j}\cdot\mathbf{A})]. \tag{2.41}$$

This expression may then be treated as a perturbation which induces an interchange of energy between the particles and the field; and in its application one meets only the same difficulties as in the Dirac dispersion theory: convergence difficulties which rest ultimately on an illegitimate application of the methods of the quantum mechanics to the electromagnetic field.

In addition to the approximation (2.39) by which, with the total neglect of retardation, one has the possibility of taking into account to all orders the Coulomb interactions of the charges, one has had, as is known, a second method of approximation by which one can take the fully retarded interaction into account to the first order only. For the case that, in the absence of interactions, the Hamiltonian of the system is even and has the characteristic values E_r, E_ρ, the interaction may to the first order be taken into account by the inclusion of a term in the energy which corresponds to the operator

$$\hat{A} = \tfrac{1}{2}\sum_{(r)(s)(l)(m)} A_{(r)(s);\,(l)(m)}\omega_{(r)(s)}\omega_{(l)(m)};$$

$$A_{(r)(s);\,(l)(m)} = \tfrac{1}{2}\int\int d\tau d\tau' \sum_{\mu=1}^{4} \frac{s^\mu_{(r)(s)}(x)s^\mu_{(l)(m)}(x')}{|r-r'|}[\exp(i\nu_{(r)(s)}r/c)+\exp(i\nu_{(l)(m)}r/c)] \tag{2.42}$$

with $\nu_{(r)(s)} = (E_{(r)} - E_{(s)})/\hbar$. But the application of this method is seriously limited on the present theory. For an unambiguous extension of this to the case where the odd parts of the Hamiltonian without interaction are not negligible does not appear to be possible, because the method presupposes that in the absence of interactions the particles remain in stationary states, and, on the present theory, this is not, in general, true; in general, namely, there will be no even states r which give determinate values to the energy of the system.

The interactions between the particles, in particular, the Coulomb interactions given by (2.40), can certainly not be entirely neglected, since one must surely expect that positives and electrons will attract and repel each other. On the other hand, there are ambiguities in the application of (2.40), which arise from the apparent impossibility of drawing a rational distinction in all cases between proper energy and interaction energy, and which we must now discuss. Two consequences in particular of these interactions must be noted. In the first place, they give rise to the possibility of the simultaneous production or destruction of more than one pair, a possibility which does not, of course, exist when they are neglected. In the second place they introduce deviations in the energy levels of an electron in a static field computed from Dirac's equation. These effects are small, and in view of the ambiguities involved in applying (2.40), it appears that the application of the formalism to these problems transcends the limits within which

the methods are valid. For these ambiguities can apparently be dissipated only by a clearer understanding of the stability of the elementary particles.

The terms in (2.40) with $C_{rs;\ lm}$, $C_{\rho\sigma;\ \lambda\mu}$ and $C_{rs;\ \rho\sigma}$ correspond to the even parts of the interactions of electrons with electrons, positives with positives, and electrons with positives. In analogy to the replacement of (2.12) by (2.14), we may eliminate the proper energies by

$$\omega_{rs}\omega_{lm}\rightarrow a_r{}^+a_l{}^+a_ma_s, \quad \omega_{\rho\sigma}\omega_{\lambda\mu}\rightarrow a_\sigma a_\mu a_\lambda{}^+a_\rho{}^+,$$

leaving

$$\omega_{rs}\omega_{\rho\sigma}=-a_r{}^+a_sa_\sigma a_\rho{}^+ \text{ and } \omega_{\rho\sigma}\omega_{rs}=-a_\sigma a_\rho{}^+a_r{}^+a_s.$$

Analogous rules can be given for all the other terms of (2.40)

$$\omega_{rs}\omega_{l\mu}\rightarrow a_r{}^+a_l{}^+a_\mu a_s, \quad \omega_{\mu r}\omega_{sl}\rightarrow a_\mu{}^+a_s{}^+a_la_r,$$
$$\omega_{\rho\sigma}\omega_{m\lambda}\rightarrow a_\sigma a_\lambda a_m{}^+a_\rho{}^+, \quad \omega_{\rho m}\omega_{\sigma\lambda}\rightarrow a_ma_\lambda a_\sigma{}^+a_\rho{}^+.$$
$$\omega_{\rho r}\omega_{s\sigma}\rightarrow a_\sigma a_\rho{}^+a_s{}^+a_r.$$

The remaining terms should not be changed. These changes have been made in such a way that no infinite contributions to the electrostatic energy appear. Because there is no equation in configuration space with which to compare our formulae, we cannot be sure that as far as the odd parts of (2.40) are concerned our results are right: they lead to no grossly incorrect consequences. In fact the distinction between proper energy and interaction energy, which has in the present state of field theory to be made, cannot be made without ambiguity in a system in which the number of particles is necessarily indeterminate.

3.

The wave functions which we have introduced, $\psi_{N,\ M}(r;\rho)$, give us directly the probability $P(r_1\cdots r_N;\rho_1\cdots\rho_M)$ of finding in the system N electrons and M positives in the states $r_1\cdots\rho_M$. Just as in the case of the general transformation functions of nonrelativistic theory, it is here not always possible to say to what extent these probabilities can be determined by actual experiment. For although the theory enables us to devise conceptual experiments for the determination of P which in no way contradict the theoretical possibilities, it is not in general possible to find actual physical means for carrying them out. This may be illustrated by considering the function ψ for a static, non-even field, and taking the (r) to be the components of momentum of the particles. Here, by suddenly destroying the field, destroying it, that is, in a time very short compared to h/mc^2, and determining in the usual way the momenta of the free particles, we can in principle determine P. Again, by scattering hard radiation, and observing the momenta of the particles of the pair and of the light after scattering, we could in principle determine the original momentum distribution with the same precision as in nonrelativistic atomic theory. But here an important point must be noted: the gamma-ray—or electron—which we use for the experiment, will itself tend to produce or destroy pairs; for the perturbation it induces is represented by an operator which is not even; and to determine the initial distribution in momentum of the pairs, we should have actually to use an "even" gamma-ray. It is clear therefore that the actual experiments in which positives are observed do not provide us with unambiguous determinations of P for the field around nuclei. It is clear too that the theory in its present form does not impose restrictions on the available experimental means, which make P unobservable; from the point of view of this theory, there could be such a thing as an "even" gamma-ray. Nor is it possible to say whether the limitations which are in fact imposed ought to be recognized by a consistent theory: whether, that is, they should be regarded as limitations in principle or by accident. It seems clear that such a limitation in the experimental possibilities could only be based on a theory in which the electromagnetic nature of the electrons and positives were understood; and it is in the sense of the quantum mechanical theory, and the approximation $\alpha\rightarrow 0$ which defines its applicability, that these restrictions should not find a counterpart in the theory, or the possibilities of observation which it offers. This situation, however, makes it particularly urgent to show how the wave functions $\psi_{N,\ M}(r;\rho)$, which are only in principle and hardly in fact observable, can be used for the calculation of things that the experiments do give us.

When the interaction of the particles can be neglected, the wave equation (2.32) determines the characteristic values of the energy of the system in static fields, and the transition probabilities in collisions and radiative processes. Now to certain of these questions, such as the energy levels of an electron in some static fields, or the probability of scattering of light by an electron, the Dirac theory of the electron gives us an answer, and one in large measure confirmed by experiment. We have now to show how, in these

cases, the new formalism gives the same answer as the Dirac theory.

Let us consider first the characteristic values of the energy of an electron in a static field. According to the Dirac theory these are determined as the characteristic values of H in the equation

$$Hu_n = E_n u_n. \tag{3.1}$$

Under certain circumstances, some of these characteristic values give correctly the energy levels of an electron in the field of force, and the corresponding wave functions may be used to describe the behavior of the electron in its stationary states. The circumstances under which this is so may be defined in the following way: if we imagine the field in which the electron is moving (or the odd part of the field) to go adiabatically to zero, the wave functions will go over into the wave functions of a free particle (or the functions for a particle in an even field of force). In the absence of degeneracies of a special kind, any solution u_n will go over into a positive or purely negative function. A function which in this sense is "genetically positive" (L) will correspond to a possible state of the electron, and the energy value E_L to a possible energy level. The functions Λ which are genetically negative do not correspond to states of the electron. Such an unambiguous application of the adiabatic theorem will, in general, only then be possible, when the stationary states corresponding to the even part of the energy are not in this sense degenerate, that positive and negative functions correspond to the same energy value. When, as in the problem of the Klein paradox, such degeneracies occur, no simple interpretation of the Dirac wave functions can be given. But in many problems, in particular in the problems of electrons in atomic fields, and in general in all stationary state problems to which the Dirac theory gives correct answers, the classification of states as genetically positive (L) or genetically negative (Λ) will be possible; and it is just in these cases that we can establish a connection between the wave functions of the present theory and those of the Dirac theory in its original form, and a corresponding equivalence of energy levels. Without in general investigating the properties of the fields for which this division of states is possible, we may remark that of the simple fields defined by a scalar potential which is a polynomial in r and $1/r$, the Coulomb field alone[8] permits this classification. As for the states Λ, which in the Dirac theory are simply rejected, we shall see that in the present theory, as we should expect, these define the states of the positive in the field, and give $-E_\Lambda$ for the corresponding energy levels.

In the present formalism the energies E_L, E_Λ, the wave functions u_L and u_Λ, do not appear directly at all. Rather (2.33) determines the states of the system when it is known that there are just as many electrons as positives present; (2.34) when it is known that there is one electron more; (2.35) when there is one extra positive, and so on. And the energies E_L must be connected with the differences in energy levels of (2.34) and (2.33), those of the positive with differences between (2.35) and (2.33). Now just in the case of fields for which, on the Dirac theory, the classification of states as L or Λ is possible, there is a unique state (O), for which the energy is a minimum, and which corresponds to the normal state of the pairs in that field. The interpretation of the wave function for this state has already been discussed; to the question of the interpretation of the energy we shall return in the next section. Let us call this energy $E^{(O)}$, and the wave functions for this state $\psi^{(O)}$. For $\nu=1$, (2.34) will give us a set of energies $E^{(L)}$ and corresponding wave functions; for $\nu=-1$, we shall get the energies $E^{(\Lambda)}$. We shall now show that the differences between the $E^{(L)}$ and $E^{(O)}$ are just the L energy values given us by the Dirac equation:

$$E^{(L)} - E^{(O)} = E_L \tag{3.2a}$$

and similarly that the differences between the $E^{(\Lambda)}$ and $E^{(O)}$ are the negative of the Λ energy values of the Dirac equation:

$$E^{(\Lambda)} - E^{(O)} = -E_\Lambda. \tag{3.2b}$$

In particular the energy differences $E^{(L)} - E^{(L')}$ are the energy differences $E_L - E_{L'}$. This result is essential, for it shows that where the Dirac theory gives correct energy levels for an electron, the present theory gives the same energies. In particular it shows that the present theory will give again Sommerfeld's formula for the relativistic doublets. In the derivation of the result it is

[8] M. S. Plesset, Phys. Rev. **41**, 278 (1932).

essential that we neglect the effect of the electromagnetic interaction of the particles of the pairs.[9]

For the proof of (3.2) one can on the one hand use perturbation methods, treating the odd parts of H as small. If we let the states r, ρ diagonalize the even part of H, we find, to the second order in $H_{r\rho}$: for $\nu=0$:

$$E^{(O)}=-\sum_{r,\rho}|H_{r\rho}|^2/(E_r-E_\rho); \tag{3.3}$$

for $\nu=1$:

$$E^{(l)}=-\sum_{r\neq l}\sum_{\rho}|H_{r\rho}|^2/(E_r-E_\rho); \tag{3.4}$$

and for $\nu=-1$

$$E^{(\lambda)}=-\sum_{r}\sum_{\rho\neq\lambda}|H_{r\rho}|^2(E_r-E_\rho). \tag{3.5}$$

The differences between (3.4) and (3.3) on the one hand, and (3.3) and (3.5) on the other, are however just the terms of this order in E_L and E_Λ for those states L and Λ which go over into l, λ when the odd part of H vanishes. In interpretation of this result it may be remarked that that contribution to the energy which in the Dirac theory may formally be ascribed to transitions to states of negative kinetic energy, here appears because of the interdiction, on introduction of the electron, of the formation of such pairs as have electrons in the state in which the electron was introduced. This perturbation method can be extended to prove (3.2) in all orders, but soon becomes unwieldy.

We may obtain a more powerful method of solving the wave equations (2.32) by the following construction. We introduce the components of the Dirac wave functions $u_{(L)}$ in the (r) space:

$$C_{(r)}{}^{(L)}=\int d\tau\bar{u}_{(r)}u_{(L)} \tag{3.6}$$

and form the wave functions

$$\begin{aligned}\psi^{((L);\ O)}_{N+1,\ N}(r;\rho)&=(N+1)^{-\frac{1}{2}}\sum_{i=1}^{N+1}(-)^{i-1}C_{ri}{}^{(L)}\psi^{(O)}{}_{N,\ N}(\cdots r_{i-1}r_{i+1}\cdots;\rho)\\&\qquad+(N+1)^{\frac{1}{2}}\sum_{\rho'}C_{\rho'}{}^{(L)}\psi^{(O)}_{N+1,\ N+1}(r;\cdots\rho_N\rho'),\\ \psi^{(O;\ (L))}_{N,\ N+1}(r;\rho)&=(N+1)^{-\frac{1}{2}}\sum_{i=1}^{N+1}(-)^{N+1-i}\bar{C}_{\rho i}{}^{(L)}\psi^{(O)}{}_{N,\ N}(r;\cdots\rho_{i-1}\rho_{i+1}\cdots)\\&\qquad-(N+1)^{\frac{1}{2}}\sum_{r'}\bar{C}_{r'}{}^{(L)}\psi^{(O)}_{N+1,\ N+1}(r'r_1\cdots;\rho).\end{aligned} \tag{3.7}$$

It may be shown then that $\psi^{((L);\ O)}$ and $\psi^{(O;\ (L))}$ satisfy (2.34) and (2.35), respectively, with energy values respectively $E^{(O)}+E_{(L)}$ and $E^{(O)}-E_{(L)}$. The proof is straightforward and gives us, for cases to which the Dirac equation is directly applicable, a method of constructing the wave functions of the present theory. Here all the solutions of the Dirac equation appear: the solutions L give us the wave functions for an electron, the solutions Λ those for a positive.

We must of course demand that

$$\psi^{(\Lambda;\ O)}_{N+1,\ N}=0;\quad \psi^{(O;\ L)}_{N,\ N+1}=0, \tag{3.8}$$

for these functions would correspond to the addition of an electron in a genetically negative state or a positive corresponding to a genetically positive state. It is easy to see by direct inspection that (3.8) must be satisfied; and we obtain a proof by constructing the $\psi^{(O)}{}_{N,\ N}$ in terms of the $C^{(L)}{}_{(r)}$. This construction of the $\psi^{(O)}{}_{N,\ N}$ is suggested by the original formulation which Dirac gave to the present theory, according to which "all negative energy states" should normally be full. Alternatively we may characterize the state (O) by saying that all the "negative energy states" of the positive are full. If then we are given a complete set of solutions u_L and u_Λ and form the quantities $C^L{}_{(r)}$, $C^\Lambda{}_{(r)}$ for all these states, we may write the solutions as infinite determinants, as follows:

$$\begin{aligned}\psi^{(O)}{}_{0,\ 0}&=|\bar{C}_r{}^L|;\\ \psi^{(O)}{}_{1,\ 1}(r_1;\rho_1)&=|\bar{C}_{r_1}{}^L\to\bar{C}_{\rho_1}{}^L|,\end{aligned} \tag{3.9}$$

[9] Cf. Section 4.

and generally

$$\psi^{(O)}{}_{N,\,N}(r_1\cdots r_N;\,\rho_1\cdots\rho_N) = (N!)^{-1}\,|\,\bar{C}_{r_i}{}^{L}\to\bar{C}^{L}{}_{\rho_{N+1-i}}\,|\,, \tag{3.9}$$

where, in these determinants, in all the rows L the $\bar{C}_{r_i}{}^{L}$ are replaced by the $\bar{C}^{L}{}_{\rho_{N+1-i}}$; here r's and ρ's should be ordered. Alternatively we may write

$$\psi^{(O)}{}_{O,\,O} = |\,C_{\rho}{}^{\Lambda}\,|\,;\quad \psi^{(O)}{}_{1,\,1}(r_1;\,\rho_1) = -\,|\,C_{\rho_1}{}^{\Lambda}\to C_{r_1}{}^{\Lambda}\,|\,, \tag{3.10}$$

$$\psi^{(O)}{}_{N,\,N}(r_1\cdots r_N;\,\rho_1\cdots\rho_N) = (-)^N(N!)^{-1}\,|\,C_{\rho_i}{}^{\Lambda}\to C^{\Lambda}{}_{r_{N+1-i}}\,|\,.$$

The $\psi^{(O)}$ so defined, if we avoid questions introduced by the convergence of the determinants, formally satisfies (2.32), with the energy value

$$E^{(O)} = -\sum_{r,\,\rho} H_{\rho r}\psi^{(O)}{}_{1,\,1}(r;\,\rho)/\psi^{(O)}{}_{O,\,O}. \tag{3.11}$$

From these expressions for $\psi^{(O)}$ the vanishing of $\psi^{(O;\,L)}$ and $\psi^{(\Lambda;\,O)}$ follows.

We may now generalize (3.7) to give us the wave functions which correspond to the addition of n electrons in the states $L_1\cdots L_n$ and m positives in the states $\Lambda_1\cdots\Lambda_m$. These functions vanish unless all the L's and all the Λ's are distinct:

$$\begin{aligned}
\psi_{N,\,M}^{(L_1\cdots L_n(L');\ \Lambda_1\cdots\Lambda_m)}(r_1\cdots r_N;\,\rho_1\cdots\rho_M) &= N^{-\frac{1}{2}}\sum_{i=1}^{N}(-1)^{i-1}C_{r_i}{}^{(L')}\psi_{N-1,\,M}^{(L_1\cdots L_n;\ \Lambda)}(\cdots r_{i-1}r_{i+1}\cdots;\,\rho)\\
&\quad+(-)^{n+m}(M+1)^{\frac{1}{2}}\sum_{\rho'}C_{\rho'}{}^{(L')}\psi_{N,\,M+1}^{(L_1\cdots L_n;\ \Lambda)}(r;\,\cdots\rho_M\rho')\\
&= 0 \text{ unless } N-M=n+1-m;\\
\psi_{NM}^{(L_1\cdots L_n;\ \Lambda_1\cdots\Lambda_m(L'))}(r_1\cdots r_N;\,\rho_1\cdots\rho_M) &= M^{-\frac{1}{2}}\sum_{i=1}^{M}(-)^{M-i}\bar{C}_{\rho_i}{}^{(L')}\psi_{N,\,M-1}^{(L;\ \Lambda_1\cdots\Lambda_m)}(r;\,\cdots\rho_{i-1}\rho_{i+1}\cdots)\\
&\quad-(-)^{n+m}(N+1)^{\frac{1}{2}}\sum_{r'}\bar{C}_{r'}{}^{(L')}\psi_{N+1,\,M}^{(L;\ \Lambda_1\cdots\Lambda_m)}(r'r_1\cdots;\,\rho)\\
&= 0 \text{ unless } N-M=n-m-1.
\end{aligned} \tag{3.12}$$

The identities corresponding to (3.8) are now of the form

$$\psi^{(L_1\cdots L_n\Lambda';\ \Lambda_1\cdots\Lambda_m)} = 0;\qquad \psi^{(L_1\cdots L_n;\ \Lambda_1\cdots\Lambda_m L')} = 0 \tag{3.13}$$

and follow, as before, from (3.9). In this way we can construct a series of solutions, corresponding to different values of n, for all of which $n=m$ and $\nu=0$. The transition from a state n to $n+1$ corresponds to the creation of a pair; that from n to $n-1$ to its annihilation. Similarly the transition, with $m=0$, $n=1$, from the state L to L' gives just the transition of the electron from one "genetically positive" state to another, and that from Λ to Λ', with $m=1$, $n=0$, gives the corresponding transition for the positives.

With the help of the functions (3.12) we can establish another useful result. Let us consider an operator of the form (2.5). To this there will correspond an operator $\hat{\omega}$ on the $\psi(N_r;\,M_\rho)$, and from this, according to (2.31), we can construct the corresponding operator as a matrix in the scheme L, Λ. The only elements of this matrix which do not vanish are those for which $(L)\to(L')$, $(\Lambda)\to(\Lambda')$, $(L;\,\Lambda)\to(O)$, or $(O)\to(L;\,\Lambda)$. By using the functions (3.12) we can now prove that

$$\begin{aligned}
(L\,|\,\hat{\omega}\,|\,L') &= (\bar{\psi}^{(L)}\cdot\hat{\omega}\psi^{(L')}) = \Omega_{LL'} = \int d\tau\bar{u}_L\Omega u_{L'};\\
(\Lambda\,|\,\hat{\omega}\,|\,\Lambda') &= (\bar{\psi}^{(\Lambda)}\cdot\hat{\omega}\psi^{(\Lambda')}) = -\Omega_{\Lambda'\Lambda} = -\int d\tau\bar{u}_{\Lambda'}\Omega u_{\Lambda};\\
(L;\,\Lambda\,|\,\hat{\omega}\,|\,O) &= (\bar{\psi}^{(L;\ \Lambda)}\cdot\hat{\omega}\psi^{(O)}) = -\Omega_{L\Lambda} = -\int d\tau\bar{u}_L\Omega u_{\Lambda};\\
(O\,|\,\hat{\omega}\,|\,L;\,\Lambda) &= (\bar{\psi}^{(O)}\cdot\hat{\omega}\psi^{(L;\ \Lambda)}) = -\Omega_{\Lambda L} = -\int d\tau\bar{u}_{\Lambda}\Omega u_{L}.
\end{aligned} \tag{3.14}$$

In this proof one has to make repeated use of the identities (3.13). For operators of the form (2.21) we may thus use the Dirac functions u_L and u_Λ to compute the matrix elements for states (L), (Λ), (O), and $(L;\Lambda)$. In general $\hat{\omega}\psi^{(L_1\cdots L_n;\ \Lambda_1\cdots\Lambda_m)}$ gives us wave functions $\psi^{(L_1\cdots L_{n-1}L';\ \Lambda_1\cdots\Lambda_m)}$, $\psi^{(L_1\cdots L_n;\ \Lambda_1\cdots\Lambda_{m-1}\Lambda')}$, $\psi^{(L_1\cdots L_nL';\ \Lambda_1\cdots\Lambda_m\Lambda')}$, and $\psi^{(L_1\cdots L_{n-1};\ \Lambda_1\cdots\Lambda_{m-1})}$. In constructing the corresponding matrix elements of $\hat{\omega}$, the antisymmetry of the ψ's in all L and all Λ must be considered.

Let us now consider transitions induced in the system by perturbations. For the present argument it does not matter whether these perturbations arise from the coupling of the system with another system, such as the radiation field, or by a direct introduction of time dependent forces. In either case, to a transition of the system in which the energy of the system changes by ΔE there will correspond a perturbing term in the Hamiltonian of the system which is of the form (2.5), and is given by the matrix $r_{(r)(s)}\exp(i\Delta Et/\hbar)$. The perturbation theory then tells us, that, to the first orders in v, and with suitable normalization, the transition probability $L\to L'$ induced by the perturbation will be

$$P_{(L)\to(L')}=(1/\hbar^2)\,|\,(L'\,|\,v\,|\,L)\,|^2 \qquad (3.15)$$

with $E^{(L')}-E^{(L)}$ necessarily equal to ΔE. Similarly

$$P_{(O)\to(L;\ \Lambda)}=(1/\hbar^2)\,|\,(L\Lambda\,|\,v\,|\,O)\,|^2;$$
$$E^{(L;\ \Lambda)}-E^{(O)}=\Delta E.$$

From (3.14) we see that to this order the transition probabilities are given by

$$\left.\begin{aligned} P_{(L)\to(L')}&=(1/\hbar^2)\,|\,v_{L'L}\,|^2 \qquad (a);\\ P_{(\Lambda)\to(\Lambda')}&=(1/\hbar^2)\,|\,v_{\Lambda\Lambda'}\,|^2 \qquad (b);\end{aligned}\right\} \qquad (3.16)$$

$$\left.\begin{aligned} P_{(O)\to(L;\ \Lambda)}&=(1/\hbar^2)\,|\,v_{L\Lambda}\,|^2 \qquad (a);\\ P_{(L;\ \Lambda)\to O}&=(1/\hbar^2)\,|\,v_{\Lambda L}\,|^2 \qquad (b).\end{aligned}\right\} \qquad (3.17)$$

This result shows that in the problems where the Dirac theory may be applied to determine the stationary states of the system, the first order transition probabilities, which give the probabilities of radiative and collision transitions, may be computed with the help of the Dirac wave functions $u_{(L)}$ just as in nonrelativistic theory. For these cases (3.16b) also shows that analogous methods may be applied to positives, and gives (3.17a) for the probability of pair production; it is this formula which is applied in computing the production of pairs by gamma-rays and electrons.

This result cannot be generally extended to the calculation of higher order transition probabilities, as is at once clear when we remember that a system which, when unperturbed, may very well admit the application of the Dirac equation, may, under the influence of the perturbation, cease to have this property. Thus the free electron or the electron in a Coulomb field may as is well known be treated by the Dirac equation; but when this system is coupled to the radiation field, the catastrophic transitions to negative energy states occur, and we cannot expect that a theory in which these transitions cannot be unambiguously excluded will give correct results. One can show, however, that to the second order in v, for any perturbation whatever, the direct use of the wave functions u_L, u_Λ and their corresponding energy values E_L, E_Λ gives, upon application of the usual perturbation methods, the correct result. For the proof of this the antisymmetry of the $\psi^{(L\cdots;\ \Lambda\cdots)}$ in the L's and Λ's is essential.

This result is sufficiently general to validate all the calculations made on the basis of the Dirac equation which have been compared with experiment. In particular it furnishes a generalization of Dirac's proof that "filling the states of negative energy" would not affect the scattering formulae of Thomson and Klein-Nishina. As far as perturbations induced by coupling an electron with the radiation field are concerned, calculations of transition to orders higher than the second cannot in any case be made, because in these higher orders the difficulties of the magnetic proper energy make an insurmountable ambiguity. For this magnetic proper energy the present theory, of course, gives new results; but these in no way remove the familiar difficulties of the radiation theory.

The limitation on the nature of the system which makes possible an unambiguous application of Dirac's equation, and which has played so large a part in the arguments of this section, admits a very simple interpretation. For this limitation means only that in such systems no genuine transitions to states of negative energy should, on the Dirac theory, be possible; on the

present theory these are the systems in which, though pairs may be present, their presence may be unambiguously detected only by disturbing the system with high energy radiations. To problems in which pairs are constantly being created by the fields present, the Dirac equation in its original form cannot be simply applied.

Throughout this section we have consistently neglected the interaction of the particles with each other, and supposed them to be affected only by given external fields. When these interactions are considered, the proofs which we have given of the applicability of the Dirac equation no longer hold, for in the original Dirac theory these interactions have of course no analogue. In particular, when these interactions are considered, one can no longer establish the rigorous validity of the scattering formulae, and the energy levels of electrons in static fields cease to be given correctly by the Dirac equation. The effects of these interactions are in general small and cannot, as we have seen, be unambiguously formulated in all cases. But within the limits here discussed, which from a physical point of view seem inevitable, we have shown that the Dirac equation, wherever it can consistently be applied, leads to the same results as the present theory. For such problems, and within these limits, it is thus not in general necessary to use the wave functions $\psi_{N,\,M}(r;\rho)$ at all; and this is fortunate, since the wave equations which determine them are in general intractable.

4.

In this section we have to consider some of the applications of the theory which have no analogue in the Dirac theory of the electron in its original form. We shall make no attempt to treat these exhaustively, but try only in a few simple cases to see how the results may be interpreted, and in what way the application of the theory is limited.

Let us look first at $E^{(0)}$. In a system in which directly observable pairs are not being created and in which therefore the solutions of Dirac's equations may be classified as L or Λ, there exists as we have seen a unique normal state for the pairs; the energy of this state is $E^{(0)}$ and is given by (3.11). This is the energy of these "nascent" pairs in the given field; when the field vanishes $E^{(0)}\rightarrow 0$; when the field does not vanish $E^{(0)}$ is in general negative. To establish the field, therefore, we do not in general have to do quite as much work as electromagnetic theory predicts; for in addition to the electromagnetic energy

$$E_e=(1/8\pi)\int d\tau(\mathfrak{E}^2+\mathfrak{H}^2) \tag{4.1}$$

in the electromagnetic field, there is an energy $E^{(0)}$ in the pairs formed by the field. This is in no sense a discrepancy with Maxwell's theory, according to which E_e should give the energy in empty space of the field $\mathfrak{E}$, $\mathfrak{H}$; for, as we have seen, the field will not be empty. This then is the simplest general interpretation of $E^{(0)}$.

If the field is itself produced by a charged particle—and here we shall assume that this particle is not an electron, but is so heavy that the reaction of the pairs upon it may be neglected—then the corresponding energy $E^{(0)}$ must be interpreted as proper energy of the particle. The only way in which $E^{(0)}$ here could be directly observable is by the creation or destruction of the particle; it could be inferred from the mass of the particle if the specifically electromagnetic energy of the particle could be computed in a satisfactory manner. For a point charge with a Coulomb field this proper energy $E^{(0)}$ is infinite and negative. It does not appear to be possible to interpret the difference between the electromagnetic and the pair energy of the particle in such a way that this difference should correspond to a finite mass; and in general it may be said that the present theory, which presupposes the particulate nature of the electron and positive, throws absolutely no light on their ultimate stability.

When more than one heavy charged particle produces the field, the energy $E^{(0)}$ will depend upon their relative positions. This dependence of the energy on the positions of the particles will correspond to a new term in the potential between them: i.e., to new forces. These forces will not in general be Coulomb forces and the deviations may be computed, and should ultimately be observable.

To make these considerations quantitative, let us consider the case of an electrostatic field, for which

$$\mathbf{A}=0; \qquad \mathfrak{E}=-\text{grad } V. \tag{4.2}$$

When the field strength is not too great, we should

expect to be able to solve (2.33) and compute $\psi^{(O)}$, by perpurbation methods, and use (3.3) for $E^{(O)}$. A sufficient condition on the field strength is that it should always be small compared to m^2c^3/eh. In this case it will be permissible to neglect entirely the even part of V which can give effects only in higher order and corresponds to the nonuniform motion of the pairs when they are accelerated by the field. Then we find

$$E^{(O)} = -\sum_{r,\,\rho} |V_{r\rho}|^2/(E_r - E_\rho), \tag{4.3}$$

where now r, ρ index the positive and negative states of a free electron. The energy density in the pairs is of the order α compared to the electromagnetic energy. For the case that the potential varies relatively little in regions of the order of the Compton wave-length we find for the ratio of the two energy densities:

$$E^{(O)}/E_e = -\alpha\kappa. \tag{4.4}$$

Here κ is given by the integral

$$\kappa = (2/\pi)\int_0^\infty k^2(1+k^2)^{-5/2}dk + (4/3\pi)\int_0^\infty k^4(1+k^2)^{-5/2}dk, \tag{4.5}$$

where in turn k is $(1/2mc)$ times the magnitude of the vector difference of the momenta of the two particles of a pair. The second integral diverges for large values of k: the contribution of pairs of large momentum does not fall off rapidly enough to give a determinate result. This divergence does not depend upon the special simplifications introduced into the problem and arises from a genuine limitation of the present theory.

This limitation is of the same kind as that met in all applications of the quantum mechanical formalism to wave fields, and comes in the divergence of κ to a similar expression; it has its analogue in the now familiar divergence difficulties of dispersion theory which may be schematically formulated as the failure of such theories when applied to extremely small lengths or intervals of time. The difficulty here has nothing to do with the introduction of point singularities in the field; the electromagnetic field we are studying is slowly varying and nowhere infinite. An analogous failure of field theory in a problem in which no singularities occur is known in gravitational theory, where the quantum theoretic calculation of the gravitational energy of an electromagnetic wave gives a divergent result.[10] These difficulties thus tend to appear in all problems in which extremely small lengths are involved; the critical lengths are presumably of the order of the cla s sical electron radius. The difficulties are of such a character that they are apparently not to be overcome merely by modifying the electromagnetic field of an electron within these small distances, but require here a more profound change in our notions of space and time, on which ultimately the quantum mechanical methods rest, and which in turn require the existence of stable particles for their definition. At present neither the precise point at which the theory breaks down nor the nature of the needed modifications can be determined; but it is clear that the sort of thing we must do with expressions like (4.5) is to break off the integral as it stands at some upper limit K corresponding roughly to the critical length e^2/mc^2, and trust a future theory to show that the contributions from k values greater than this are small. It is from this at once apparent that the theory in its present form can make no predictions whatever about the fields within the critical distance e^2/mc^2 of a charge.

In the case of the production of pairs, there is direct evidence that the present theory gives too large a probability for high energy pairs. For the probability of production of pairs by a beam of gamma-rays in the field of nuclei, when computed on the basis of the present theory, turns out to be much larger than any value which could be reconciled with the known penetrating power of high energy gamma-rays. And in this case too the failure of the theory seems in no way connected with the magnitude of the nuclear field in the immediate neighborhood of the nuclei. The fact that the theory should fail here is not very surprising when we remember that even on classical theory the model of a point electron, which underlies the present theory, would give altogether wrong results for the reaction of the electron to light of wave-length appreciably shorter than the critical length e^2/mc^2.

With this understanding, then, κ is given by

[10] L. Rosenfeld, Zeits. f. Physik **65**, 589 (1930).

$(4/3\pi)\ln 2K - 10/9\pi$ and is about 2 with $K\sim 1/\alpha$. The result then tells us that the work we must do to establish an electrostatic field is about 2 percent less than the energy stored in the electromagnetic field; the difference is supplied by the pairs. One might at first suppose that this result would have catastrophic consequences for electromagnetic theory. But one sees at once that the forces exerted upon a point charge (by a field which still varies little within λ_0), which serve to define the field strengths in terms of the unit of charge, are reduced by the pairs formed about the charge in just the ratio (4.4). And we may readily see that the forces exerted by two charges upon each other (again at distances large compared to λ_0), which serve to define the unit of charge, are similarly reduced; so that as long as we do not consider lengths smaller than λ_0—which vanishes as h goes to zero as the correspondence principle requires—the consistency of classical electromagnetic theory is not affected at all, and the difference between the "true" charges, and "true" fields and those which include the effect of the pairs, are not, under these circumstances, observable.[10a]

We may calculate simply this dependence of pair energy on the separation of two charges. If the charge on the particles is not greater than Ze, then the condition $|\mathfrak{C}| < m^2c^3/he$ means only that the particles shall stay far apart compared to $(\alpha Z)^{\frac{1}{3}}\lambda_0$. If the two particles are separated by a distance R, then

$$E^{(0)}_{\text{int.}} = -(\alpha e_1 e_2/4\pi^4\lambda_0)\int\int d\mathbf{k}d\mathbf{k}' \frac{[\epsilon\epsilon' - (\mathbf{k}\cdot\mathbf{k}') - 1]e^{i(\mathbf{k}-\mathbf{k}')\cdot R/\lambda_0}}{\epsilon\epsilon'(\epsilon+\epsilon')\,|k-k'|^4} \tag{4.6}$$

with $\mathbf{k}$, $\mathbf{k}' = (2\pi/mc)$ times the momentum of the positive and electron, and $\epsilon = (1+k^2)^{\frac{1}{2}}$. When $R \gg \lambda_0$, this may be evaluated and gives just

$$E^{(0)}_{\text{int.}} = -\alpha\kappa e_1 e_2/R. \tag{4.7}$$

This result shows that if we define the unit charge[11] in the usual way and the field strengths in terms of this charge, then the energy in the field will be given by E_e of electromagnetic theory. In situations where the charges are not strongly accelerated and in which no lengths small compared to λ_0 enter, the difference between the charge so defined and the true charge Ze, will not be observable.[12] In principle a detection of this difference between true and apparent charge could be expected for a violently accelerated electron; the effects would be of the order α for accelerations of the order mc^3/h. But for such accelerations the electron theoretic behavior of an electron is uncertain by effects of just the order α which is the ratio of the acceleration to mc^4/e^2. Although, therefore, the effects here discussed would in principle be capable of observation if we had an adequate electromagnetic theory with which to compare them and would in that case have to be taken into account in describing the behavior of the electron, they would appear in fact in the present state of the theory not to be unambiguously detectable by observation.

A similar argument shows that the deviations from (4.7) given by (4.6) for the field at distances $R < \lambda_0$ from an electron are also in this sense unobservable, that they are masked by the

[10a] *Note added in proof:* Because of the gauge dependence of the theory no unambiguous calculation of the polarization energy of the pairs in a magnetostatic field can be made; and in fact no straightforward choice of gauge gives for this energy that value formally equal to (4.4), which alone is consistent with relativity. This consideration shows that, quite apart from the ambiguities discussed above, the present calculations of the reaction of the pairs on the field may have to be essentially modified.

[11] Through the kindness of Professor Lawrence, we have just seen Dirac's report to the Solvay Congress. In this report, Dirac, from a different point of view, is led to a result that the charge of an electron defined in the usual way is not the true charge. The expression which Dirac derives for the difference between these charges is in complete agreement with (4.7).

[12] This can be said in another way: Because of the polarizability of the nascent pairs, the dielectric constant of space into which no matter has been introduced differs from that of truly empty space. For fields which are neither too strong nor too rapidly varying the dielectric constant of a vacuum then has the constant value $\sim(1+\kappa\alpha)$. Because it is in practice impossible not to have pairs present, we may redefine all dielectric constants, as is customarily done, by taking that of a vacuum to be unity.

electron theoretic uncertainties in the behavior of an accelerated electron. Thus these deviations in the field give a shift in the energy levels of an atomic electron predicted by the Dirac theory; but these shifts are never greater than the uncertainties in the energy levels which arise from our ignorance of the reaction of the electron to its own radiation field.

For protons, for which we may perhaps suppose that the electron theoretic ambiguities arise only for the far larger accelerations Mc^4/e^2 the deviations from the Coulomb law should in principle be detectable. These deviations arise at distances $R\sim\lambda_0$. For $R\ll\lambda_0$, but still, in order that KR may be regarded as large, large compared to e^2/nc^2, (4.6) gives:

$$E_{\text{int.}}^{(0)} = -(\alpha e^2/R)[\kappa+((10/9)\pi) +((4/3)\pi)(\ln(2\pi R/\lambda_0)+\gamma)] \quad (4.8)$$

with γ = Euler's constant. This corresponds to an increase of effective charge of the proton of

$$\delta e\sim(2\alpha e/3\pi)\ln(\lambda_0/25R). \quad (4.9)$$

The effect is small, but may perhaps be detected by scattering experiments. For the scattering of protons by protons, deviations from the scattering for a Coulomb field of the order of one percent should appear for proton energies of a hundred thousand volts. It must again be emphasized that for very close distances of approach, where these deviations may be large, the theory may not be applied; for the methods here used to give a meaning to κ must rather be thought of as the first steps in an approximation based upon the actual smallness of α.