An introduction to QED

- Particle-antiparticle
- Relativistic propagator
- Feynman–Kac formula
- Dyson series
- Feynman diagrams
- Running coupling constant α

The Dirac equation (*a contemporary point of view*)

Let $\psi \in \mathbb{C}(\mathbb{R}^4)$. The Klein–Gordon equation is

$$\partial^2\psi = m^2\psi \qquad\qquad \partial^2 = -\partial_0^2 + \nabla^2$$

The current $J = i(\psi\partial\psi^* - \psi^*\partial\psi)$ is conserved, i.e. $\partial J = 0$

In 1928 Dirac obtained a decomposition using the Pauli matrices

$$i(\partial_0 + \sigma_i\partial_i)u = mv$$
$$i(\partial_0 - \sigma_i\partial_i)v = mu$$

v is the mirror image of u, in the sense $x_i \mapsto -x_i$

Irreducible representation of $SO^+(1,3) = SL(2,\mathbb{C})/\mathbb{Z}_2$

$E = i\sigma_2$ is the metric on the linear space of two-component spinors, Λ is a 2×2 complex matrix, s.t. $\det\Lambda = 1$. Then $\xi^\nu E_{\nu\mu}\eta^\mu = \det\|\xi\eta\|$ is invariant under the transformation $\xi' = \Lambda\xi$, $\eta' = \Lambda\eta$

$$\|\xi\eta\| = \begin{pmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{pmatrix} \qquad \det\|\xi'\eta'\| = \det(\Lambda\|\xi\eta\|) = \det\Lambda\det\|\xi\eta\|$$

In the space of 2×2 anti-hermitian matrices $M^{\lambda\dot{\mu}}$

$$M = i\begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} = i(x^0 + x^i\sigma_i)$$

the transformation $M \to \Lambda M \Lambda^+$ furnishes an irreducible representation of the proper orthochronous Lorentz group, $\det M = x^\mu g_{\mu\nu} x^\nu$. Here x^μ is a contravariant vector in Minkowski space, $g_{\mu\nu} = (-1,1,1,1)$.

Particle-antiparticle

$$i\partial_{\dot{\mu}\lambda}u^{\lambda} - mv_{\dot{\mu}} = 0 \qquad\qquad i\partial^{\lambda\dot{\mu}}v_{\dot{\mu}} - mu^{\lambda} = 0 \qquad (1)$$

Let $w^{\mu} = v^{*\mu}$. $U(1)$ gauge invariant equations:

$$i(\partial_{\dot{\mu}\lambda} - ieA_{\dot{\mu}\lambda})u^{\lambda} - mw^{*}_{\dot{\mu}} = 0 \qquad i(\partial_{\dot{\mu}\lambda} + ieA_{\dot{\mu}\lambda})w^{\lambda} - mu^{*}_{\dot{\mu}} = 0$$

Put $A = 0$ and $u_t = e^{-iEt}u_0$

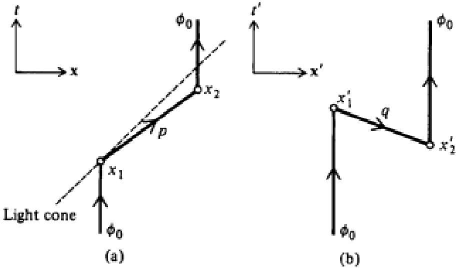
$$(u_t, w_t) = \begin{cases} (e^{-i\omega t}u_0^{(+)}, e^{+i\omega t}w_0^{(+)}) & E > 0 \quad (1) \\ (e^{+i\omega t}u_0^{(-)}, e^{-i\omega t}w_0^{(-)}) & E < 0 \quad (2) \end{cases} \qquad \omega = |E|$$

Furthermore put $b_{\dot{\mu}} = u^{*}_{\dot{\mu}}$

$$i\partial_{\dot{\mu}\lambda}w^{\lambda} - mb_{\dot{\mu}} = 0 \qquad\qquad i\partial^{\lambda\dot{\mu}}b_{\dot{\mu}} - mw^{\lambda} = 0 \qquad (2)$$

Since v is the charge conjugate of w it is natural to identify u and v with the chiral components of the particle $\psi = (u^{\lambda}, v_{\dot{\mu}})$

The Feynman propagator



For a virtual particle evolution of state during a time interval $\delta t = t_2 - t_1$ is

$$\phi = \begin{cases} e^{-iE\delta t}\phi_1 & \delta t > 0, \quad E > 0 \quad (a) \\ -e^{-i(-E)|\delta t|}\phi_2 & \delta t < 0, \quad E < 0 \quad (b) \end{cases}$$

The case $\delta t < 0$ corresponds to the propagation $\phi_2 \to \phi_1$ of the antiparticle with a positive energy $-E$.

Feynman–Kac formula

The Feynman propagator in Minkowski space:

$$-i\Delta_F(x,y) = \theta(x^0 - y^0)P_+ - \theta(y^0 - x^0)P_-$$

Green's function of the Dirac equation:

$$(\partial\!\!\!/ + m)S = \delta^4 \qquad S = \Delta_F \beta \qquad \beta = i\gamma^0 \qquad \partial\!\!\!/ = \gamma^\mu \partial_\mu$$

For $t > 0$ we go into Euclidean space $t \mapsto -it$, $2n$-point correlator:

$$\langle \psi_{x_1} \dots \psi_{x_n} \bar{\psi}_{y_1} \dots \bar{\psi}_{y_n} \rangle = \int\limits_{\mathcal{D}\bar{\psi}\mathcal{D}\psi} \psi_{x_1} \dots \psi_{x_n} \bar{\psi}_{y_1} \dots \bar{\psi}_{y_n} \, e^{-\int\limits_z \bar{\psi}_z S^{-1} \psi_z}$$

$$S_{xy} = \langle \psi_x \bar{\psi}_y \rangle = -\frac{\delta^2}{\delta\bar{\eta}_x \delta\eta_y} \int\limits_{\mathcal{D}\bar{\psi}\mathcal{D}\psi} e^{-\int\limits_z \bar{\psi}_z S^{-1} \psi_z + \int\limits_z \bar{\eta}_z \psi_z + \int\limits_z \bar{\psi}_z \eta_z} \Bigg|_{\substack{\eta=0 \\ \bar{\eta}=0}}$$

Dyson series

Put $S^{-1} = \not{\partial} - ie\not{A} + m$ and $C^{-1} = -\partial^2$

$$Z_{\bar{\eta}\eta j} = \int\limits_{\mathcal{D}A\mathcal{D}\bar{\psi}\mathcal{D}\psi} e^{-\frac{1}{2}\int_z A_z C^{-1} A_z - \int_z \bar{\psi}_z S^{-1}\psi_z + \int_z \bar{\eta}_z \psi_z + \int_z \bar{\psi}_z \eta_z + \int_z j_z A_z} \; e^{ie\int_z \bar{\psi}_z \not{A}_z \psi_z}$$

Dressed 2-point function

$$\langle \psi_x \bar{\psi}_y \rangle = \sum_{n=0}^{\infty} \frac{(ie)^n}{n!} \int\limits_{z_1 \ldots z_n} \langle \psi_x \bar{\psi}_y (\bar{\psi}_{z_1} \not{A}_{z_1} \psi_{z_1}) \ldots (\bar{\psi}_{z_n} \not{A}_{z_n} \psi_{z_n}) \rangle$$

Feynman diagrams

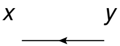


Figure: $\langle \psi_x \bar{\psi}_y \rangle$

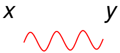


Figure: $\langle A_x A_y \rangle$

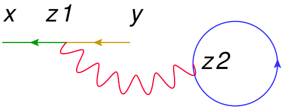


Figure: $\langle \psi_x \bar{\psi}_y \bar{\psi}_{z_1} \not{A}_{z_1} \psi_{z_1} \bar{\psi}_{z_2} \not{A}_{z_2} \psi_{z_2} \rangle$

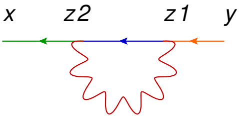
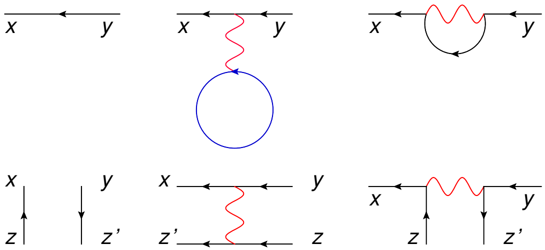


Figure: $\langle \psi_x \bar{\psi}_y \bar{\psi}_{z_1} \not{A}_{z_1} \psi_{z_1} \bar{\psi}_{z_2} \not{A}_{z_2} \psi_{z_2} \rangle$

Electron in a heavy atom (Large N approximation)

$$S^f_{xy} = \langle \Phi_N , \psi_x \, \psi^+_y \Phi_N \rangle \quad \Phi_N = \prod_{i=1}^{N} \psi^+(f_i) \quad f_i = \delta_t \varphi_i \quad \langle \varphi^+_i \varphi_j \rangle = \delta_{ij}$$



$$S^f_{xy} = S_{xy} + \sum_{i=1}^{N} S^{(i)}_{xy} \qquad S^{(i)}_{xy} = -e^{-\omega_i (x^0 - y^0)} \varphi_i(\vec{x}) \bar{\varphi}_i(\vec{y})$$

The Hamiltonian formalism

Using the Fourier transform of the time we have

$$(\alpha^k(\partial_k - ie\mathcal{A}_k) - e\mathcal{A}_0 + \beta m - \omega)\bar{S}(\omega) = \delta^3 \qquad \bar{S} = S\beta$$

To calculate the resolvent $\bar{S}$ we use the bilinear formula, i.e.

$$H\psi_\lambda = \lambda\psi_\lambda$$

$$\bar{S}_{\vec{x}\vec{y}}(w) = \int d\lambda \, \frac{\psi_\lambda(\vec{x})\psi_\lambda^+(\vec{y})}{\lambda_\mp - \omega} \qquad \lambda_\mp = \begin{cases} \lambda - i\epsilon & \lambda > 0 \\ \lambda + i\epsilon & \lambda < 0 \end{cases}$$

The position of the poles has been chosen such that $\lim_{t \to 0\pm} -iS(t)\beta = \pm P_\pm$

$$i\bar{S}_{\vec{x}\vec{y}}\Big|_{t=0} = i\int \frac{d\omega}{2\pi}\bar{S}_{\vec{x}\vec{y}}(\omega) = \frac{1}{2}R_1 \qquad R_1 = P_- - P_+$$

The effective action

$\mathcal{A}$ is a classical potential, e.g. $\mathcal{A} = (-\frac{eZ}{r}, 0)$

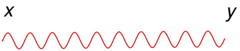
$$\Gamma = \int\limits_{xy\in\mathbb{R}^4} \bar{\psi}_x \left(\not{\partial} - ie\not{\mathcal{A}} + m + ie\not{C}\Gamma^A - \Gamma^{\bar{\psi}\psi}\right)_{xy} \psi_y$$

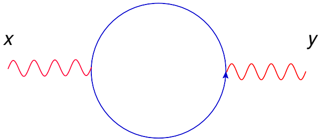
We can break the Lorentz and gauge symmetries

$$ie\not{C}\Gamma^A = \beta\delta_{xy} \int\limits_{\vec{z}\in\mathbb{R}^3} \frac{e^2 Tr(-iS\beta)_{\vec{z}} + e(Z_3 - 1)\mathcal{J}^0_{\vec{z}}}{4\pi|\vec{x} - \vec{z}|}$$

$$\Gamma^{\bar{\psi}\psi} = \alpha\beta\frac{-iS_{xy}\beta}{|\vec{x} - \vec{y}|}\delta_t + (Z_2 - 1)(\frac{4}{3}\gamma^k(\partial_k - ie\mathcal{A}_k) + 2m)\delta_{xy}$$

Vacuum polarization





$$V(r) = -\frac{\alpha(r, \alpha_0)}{r} \qquad \alpha(p, \alpha_0) = \alpha_0 + \frac{\alpha_0^2}{3\pi} \log \frac{p^2}{\mu^2} + O(\alpha_0^3)$$

Gell–Mann and Low RG equation

For $t > 0$ let $\mu_0^2 \mapsto \mu^2 = t\mu_0^2$. We have a functional equation

$$\alpha(x, \alpha_0) = \alpha(\frac{x}{t}, \alpha(t, \alpha_0)) \qquad \alpha(1, \alpha_0) = \alpha_0 \qquad x = \frac{p^2}{\mu_0^2}$$

A differential form at $t = x$

$$\frac{\partial\alpha(x, \alpha_0)}{\partial \log x} = \left(\frac{\partial}{\partial \log x} \frac{x}{t}\right) \frac{\partial\alpha(\xi, \alpha(t, \alpha_0))}{\partial\xi}\Big|_{\substack{\xi=1 \\ t=x}} = \beta(\alpha(x, \alpha_0))$$

$$\beta(\tilde{\alpha}) = \frac{\partial\alpha(\xi, \tilde{\alpha})}{\partial\xi}\Big|_{\xi=1}$$

Running coupling α

The β-function:

$$\beta(\alpha_0) = \frac{\partial \alpha(\xi, \alpha_0)}{\partial \xi}\Big|_{\xi=1} = \frac{1}{3\pi}\,\alpha_0^2 + O(\alpha_0^3)$$

The RG equation:

$$\frac{\partial \alpha(x, \alpha_0)}{\partial \log x} = \frac{1}{3\pi}\,\alpha^2(x, \alpha_0) \implies \alpha(x, \alpha_0) = \frac{1}{c - \frac{1}{3\pi}\log x}$$

The boundary condition $\alpha(1, \alpha_0) = \alpha_0$. This implies

$$\alpha(x, \alpha_0) = \frac{\alpha_0}{1 - \frac{\alpha_0}{3\pi}\log x} \qquad x = \frac{p^2}{\mu_0^2}$$