Time-dependency of vector product for central-force movement:

$$\frac{d}{dt}(\vec{p}\times\vec{\ell}) = \frac{d\vec{p}}{dt}\times\vec{\ell}$$

since we have already shown that $\frac{d\vec{\ell}}{dt} = \vec{0}$ via Noether current.

Newton's eq. of motion:

$$\frac{d\vec{p}}{dt} = \vec{F} = f(r)\,\vec{e}_r$$

one central force here.

$$\Rightarrow \frac{d}{dt}(\vec{p}\times\vec{\ell}) = f(r)\frac{1}{r}\,\vec{r}\times(\vec{r}\times\vec{p})$$

$$= \frac{m f(r)}{r}\,\vec{r}\times\left(\vec{r}\times\frac{d\vec{r}}{dt}\right)$$

$$= \frac{m f(r)}{r}\left[\left(\vec{r}\cdot\frac{d\vec{r}}{dt}\right)\vec{r} - \frac{d\vec{r}}{dt}\,\vec{r}^{\,2}\right]$$

"bac - cab"

we calculate the "auxiliary"

$$\frac{d}{dt}(\vec{r}\cdot\vec{r}) = 2\boxed{\frac{d\vec{r}}{dt}\cdot\vec{r}}$$

$$\frac{d}{dt}\vec{r}^{\,2} = 2\boxed{r\frac{dr}{dt}}$$

and so we get

$$\frac{d}{dt}(\vec{p}\times\vec{\ell}) = -\frac{m f(r)}{r}\left[r^2\frac{d\vec{r}}{dt} - \vec{r}\,r\frac{dr}{dt}\right] = -m f(r)\, r^2\left[\frac{1}{r}\frac{d\vec{r}}{dt} - \frac{\vec{r}}{r^2}\frac{dr}{dt}\right]$$

with another auxiliary we have

$$\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{1}{r}\frac{d\vec{r}}{dt} - \frac{\vec{r}}{r^2}\frac{dr}{dt}$$

$$\left[\frac{d}{dt}r^{-1} = -1\,r^{-2}\frac{dr}{dt} = -\frac{1}{r^2}\frac{dr}{dt}\right]$$

and so

$$\frac{d}{dt}(\vec{p}\times\vec{\ell}) = -m f(r)\, r^2\frac{d}{dt}\left(\frac{\vec{r}}{r}\right)$$

and with $f(r) = -\frac{\alpha}{r^2}$

$$= m\alpha\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{d}{dt}\left(m\alpha\frac{\vec{r}}{r}\right)$$

so we have found that

$$\delta\vec{\varepsilon}\cdot\frac{d}{dt}\left(\vec{p}\times\vec{\ell}\right) = \delta\vec{\varepsilon}\cdot\frac{d}{dt}\left(m\alpha\frac{\vec{r}}{r}\right)$$

$$\Leftrightarrow \delta\vec{\varepsilon}\cdot\frac{d}{dt}\left[\vec{p}\times\vec{\ell} - m\alpha\frac{\vec{r}}{r}\right] = 0$$

avec $\delta\vec{\varepsilon}\neq 0$

$$\Rightarrow \frac{d}{dt}\vec{M} := \frac{d}{dt}\left[\frac{1}{m}\,\vec{p}\times\left(\vec{r}\times\vec{p}\right) - \alpha\frac{\vec{r}}{r}\right] = \vec{0}$$

$\square$.