4 ## SO(3)

Special Orthogonal continuous group of rotations in 3 real dimensions.

Here we work backwards, knowing how to rotate a vector in a plane :

(a)

group elements

$$R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}$$ rotation in (y,z) plane

$$R_y(\psi) = \begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ +\sin\psi & 0 & \cos\psi \end{pmatrix}$$ (x,z)

$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ (x,y)

Euler matrices.

SO(3) has 3 parameters, ϕ, ψ, θ.

Now we find the generators from the definition

(b)

$$X_x = -i\left.\frac{\partial R_x(\phi)}{\partial\phi}\right|_{\phi=0} = -i\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin\phi|_{\phi=0} & \cos\phi|_{\phi=0} \\ 0 & -\cos\phi|_{\phi=0} & -\sin\phi|_{\phi=0} \end{pmatrix}$$

$$= -i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$ which is antisymmetric.

and similarly

$$X_y = -i\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad X_z = -i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

c) $|l, m_l\rangle$

$$\hat{l}_3 |l, m_l\rangle = \hbar m_l |l, m_l\rangle$$

$$\Rightarrow \langle l, m_l' | \hat{l}_3 | l, m_l\rangle = \hbar \delta_{m_l, m_l'} m_l$$

donc $X_c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$\underline{\underline{L}}_3 = \hbar X_c$

générateur Cartan de SO(3) (matrice diagonale)

(d)
$$\Rightarrow \hat{U}(\varphi) = e^{-\frac{i}{\hbar}\varphi \underline{\underline{L}}_3} = e^{-i\varphi \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - i\varphi \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} - \frac{\varphi^2}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{i}{6}\varphi^3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdots$$

$$= \begin{pmatrix} 1 - i\varphi - \frac{\varphi^2}{2} + \frac{i}{6}\varphi^3 \cdots & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 + i\varphi - \frac{\varphi^2}{2} - \frac{i}{6}\varphi^3 \cdots \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\varphi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\varphi} \end{pmatrix} =$$

$\boxed{(e)}$

$\varphi = 2\pi$

$\Rightarrow e^{i2\pi} = \cos(2\pi) + i\sin(2\pi) = 1 = e^{-i2\pi}$

(b)

$$X_z = -i \left. \frac{\partial R_z(\varphi)}{\partial \varphi} \right|_{\varphi=0}$$

$$R_z(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$X_z = -i \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$X_x = -i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & +1 & 0 \end{pmatrix}$$

$$R_x(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & +\sin\varphi & \cos\varphi \end{pmatrix}$$

$$X_y = -i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$R_y(\varphi) = \begin{pmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{pmatrix}$$

SO(3) is non-Abelian group.

The Lie-Algebra of SO(3) is identical to the algebra of angular momentum operators.

$\Rightarrow$ Angular momentum is the generator of rotations.

$\rightarrow$ We could have constructed the generators of SO(3) using an angular momentum basis

$$|l, m_l\rangle \in \{|1,1\rangle, |1,0\rangle, |1,-1\rangle\}$$

via

$$\langle l, m_l | \hat{l}_1 | l, m_l' \rangle = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = L_1$$

$$\langle l, m_l | \hat{l}_2 | l, m_l' \rangle = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} = L_2$$

$$\langle l, m_l | \hat{l}_3 | l, m_l' \rangle = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = L_3$$

which, however, satisfy the same Lie algebra as above.

(we will revisit these generators later.)

$$[X_x, X_y] = -\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = +iX_z$$

$$[X_y, X_z] = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = +i X_x$$

$$[X_z, X_x] = -\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = +i X_y$$

$$\Rightarrow \quad [X_i, X_j] = +i \sum_{k=1}^{3} \varepsilon_{ijk} X_k \quad \checkmark$$

algèbre d'un moment cinétique.

3(a) $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$ $\qquad \cosh(x) = \frac{1}{2}(e^x + e^{-x})$

$$\phi = \operatorname{arctanh}\left(\frac{v}{c}\right) \quad \Rightarrow \quad \frac{v}{c} = \tanh\phi = \frac{\sinh\phi}{\cosh\phi}$$

$$\cosh^2(x) = \frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right)$$

$$\sinh^2(x) = \frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right)$$

$$\Rightarrow \cosh^2(x) - \sinh^2(x) = 1$$

$$\Rightarrow \frac{v^2}{c^2} = \frac{\cosh^2\phi - 1}{\cosh^2\phi} \quad \Rightarrow \quad \gamma^2 = \frac{1}{1-\frac{v^2}{c^2}} = \frac{1}{1 - \frac{\cosh^2\phi - 1}{\cosh^2\phi}}$$

$$= \frac{1}{\frac{\cancel{\cosh^2\phi} - \cancel{\cosh^2\phi} + 1}{\cosh^2\phi}} = \cosh^2\phi$$

$$\gamma = \cosh\phi$$

$$\Rightarrow \frac{v}{c}\gamma = \tanh\phi \cosh\phi = \sinh\phi$$

$$\langle \tfrac{1}{2},\tfrac{1}{2}| j_2 |\tfrac{1}{2},-\tfrac{1}{2}\rangle = \frac{1}{2i}\langle \tfrac{1}{2},\tfrac{1}{2}| j_+ |\tfrac{1}{2},-\tfrac{1}{2}\rangle = \frac{1}{2i}$$

$$\langle \tfrac{1}{2},-\tfrac{1}{2}| j_2 |\tfrac{1}{2},\tfrac{1}{2}\rangle = \frac{1}{2i}\langle \tfrac{1}{2},-\tfrac{1}{2}| -j_- |\tfrac{1}{2},+\tfrac{1}{2}\rangle = -\frac{1}{2i}$$

les autres sont $=0$.

$$\Rightarrow \boxed{J_y^{(1/2)} = J_2^{(1/2)} = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\sigma_2}{2}}$$

Interprétation :

Les générateurs de la représentation $j=\frac{1}{2}$ sont identiques (sauf unité) aux matrices de Pauli.

$\Rightarrow$ spin est générateur d'une rotation dans l'espace de spin (complexe).

$$\boxed{J_z^{(1/2)} = J_3^{(1/2)} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\sigma_3}{2}}$$

2 (a)

$$\hat{S}_3 |s, m_s\rangle = \hbar m_s |s, m_s\rangle$$

$$\hat{S}_3 |\tfrac{1}{2},\tfrac{1}{2}\rangle = \frac{\hbar}{2} |\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$\hat{S}_3 |\tfrac{1}{2},-\tfrac{1}{2}\rangle = -\frac{\hbar}{2} |\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$\Rightarrow \begin{pmatrix} \langle \tfrac{1}{2},\tfrac{1}{2}|\hat{S}_3|\tfrac{1}{2},\tfrac{1}{2}\rangle = \frac{\hbar}{2} & \langle \tfrac{1}{2},\tfrac{1}{2}|\hat{S}_3|\tfrac{1}{2},-\tfrac{1}{2}\rangle = 0 \\ \langle \tfrac{1}{2},-\tfrac{1}{2}|\hat{S}_3|\tfrac{1}{2},\tfrac{1}{2}\rangle = 0 & \langle \tfrac{1}{2},-\tfrac{1}{2}|\hat{S}_3|\tfrac{1}{2},-\tfrac{1}{2}\rangle = -\frac{\hbar}{2} \end{pmatrix}$$

$$= \frac{\hbar}{2}\sigma_3$$

$$\Rightarrow \begin{pmatrix} \gamma & -\frac{v}{c}\gamma \\ -\frac{v}{c}\gamma & \gamma \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}$$

6.2 Lorentz group : rotations and boosts

4 spacetime dimensions.

We derive the generators from the transformation matrices :

Three spatial rotations

$$R_x(\Theta_x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\Theta_x & \sin\Theta_x \\ 0 & 0 & -\sin\Theta_x & \cos\Theta_x \end{pmatrix}$$

$$R_y(\Theta_y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\Theta_y & 0 & -\sin\Theta_y \\ 0 & 0 & 1 & 0 \\ 0 & \sin\Theta_y & 0 & \cos\Theta_y \end{pmatrix}$$

$$R_z(\Theta_z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\Theta_z & \sin\Theta_z & 0 \\ 0 & -\sin\Theta_z & \cos\Theta_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

SPACE-TIME VECTORS TRANSFORM UNDER ROTATIONS AS TO THE ABOVE.

So the boost matrices are written as

$$B_x(\phi_x) = \begin{pmatrix} \cosh\phi_x & -\sinh\phi_x & 0 & 0 \\ -\sinh\phi_x & \cosh\phi_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B_y(\phi_y) = \begin{pmatrix} \cosh\phi_y & 0 & -\sinh\phi_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh\phi_y & 0 & \cosh\phi_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B_z(\phi_z) = \begin{pmatrix} \cosh\phi_z & 0 & 0 & -\sinh\phi_z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\phi_z & 0 & 0 & \cosh\phi_z \end{pmatrix}$$

The generators for the rotations are analogous to those of $SO(3)$:

$$J_x = -i \left. \frac{\partial R_x(\theta_x)}{\partial \theta_x} \right|_{\theta_x=0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$J_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \qquad J_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The boost generators are found to be, using

$$\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$
$$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

$$\frac{\partial \cosh x}{\partial x} = \sinh x \qquad \frac{\partial \sinh x}{\partial x} = \cosh x$$

$$K_x = -i \left.\frac{\partial B_x(\phi_x)}{\partial \phi_x}\right|_{\phi_x = 0} = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_y = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_z = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

In order to define the Lie algebra of the Lorentz group, we need to evaluate the commutators of the generators.

Now we can determine the matrix algebra for rotations and boosts:

$$[J_i, J_j] = i \sum_{k=1}^{3} \varepsilon_{ijk} J_k$$

immediately obvious due to isomorphism on SO(3).

However, the commutator of boosts gives

e.g.

$$[K_x, K_y] = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$- \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= -i J_z$$

The commutator of two boosts is a rotation!

→ application : précession de Thomas

In general we find :

$$[K_i, K_j] = -i \sum_{k=1}^{3} \varepsilon_{ijk} J_k$$

So rotations form a closed subalgebra within the Lorentz group, whereas boosts do not.

Finally, for combined boosts and rotations :

$$[J_x, K_y] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = i K_z$$

and in general:

$$[J_i, K_j] = i \sum_{k=1}^{3} \varepsilon_{ijk} K_k$$

The general Lorentz transformation with rotations and boosts can, therefore be written as

$$L = e^{i\vec{J}\cdot\vec{\Theta} + i\vec{K}\cdot\vec{\Phi}}$$

where $\vec{J}, \vec{K}$ are 3-dim. vectors of 4×4 gener matrices and $\vec{\Theta}, \vec{\Phi}$ are 3-dim. vectors of scalar transformation parameters.

Due to the specificity of the metric (or space-time geometry), the Lorentz Lie group is denoted as

$$SO(1,3)$$

$\det(\underline{\underline{\Lambda}}) = +1$ $\underline{\underline{\Lambda}}^T = \underline{\underline{\Lambda}}^{-1}$ temps   espace

6.3 (2.2.3) Underlying structure of SO(1,3)

We want to construct closed subalgebras of SO(1,3).

This goes as follows:

$$N_i^{\pm} := \frac{1}{2}\left(J_i \pm i K_i\right)$$

Then,
$$[N_i^{\pm}, N_j^{\pm}] = \frac{1}{4}\left[(J_i \pm i K_i), (J_j \pm i K_j)\right]$$

$$= \frac{1}{4}\left\{[J_i, J_j] \pm i [J_i, K_j] \pm i [K_i, J_j] - [K_i, K_j]\right\}$$

$$= \frac{1}{4}\left\{i \sum_{k=1}^{3} \varepsilon_{ijk} J_k \pm i^2 \sum_{k=1}^{3} \varepsilon_{ijk} K_k \mp i^2 \sum_{k=1}^{3} \varepsilon_{jik} K_k + i \sum_{k=1}^{3} \varepsilon_{ijk} J_k\right\}$$

$$= \frac{1}{4}\left\{2i \sum_{k=1}^{3} \varepsilon_{ijk} J_k \pm 2i^2 \sum_{k=1}^{3} \varepsilon_{ijk} K_k\right\}$$

$$= i \sum_{k=1}^{3} \varepsilon_{ijk} \frac{1}{2}\left(J_k \pm i K_k\right)$$

$$= i \sum_{k=1}^{3} \varepsilon_{ijk} N_k^{\pm}$$

It is easily shown that

$$[N_i^{+}, N_j^{-}] = 0 \qquad \forall\, i,j \in \{1,2,3\}$$

So the sets $\{N_i^{+}\}$ and $\{N_i^{-}\}$ form <u>closed subalgebras</u>, (otherwise $[\,,\,]$ would have to be $\neq 0$) of the form of those for SU(2).

$$\Rightarrow \quad SO(1,3) = SU(2) \circledast SU(2)$$

<u>in this representation</u> (two copies of SU(2)).