$$\left[-\begin{pmatrix} \mathbb{1}_2 & 0_2 \\ 0_2 & -\mathbb{1}_2 \end{pmatrix} \imath\hbar \frac{\partial}{\partial t} + c \begin{pmatrix} 0_2 & \boldsymbol{\sigma}\cdot\mathbf{p} \\ -\boldsymbol{\sigma}\cdot\mathbf{p} & 0_2 \end{pmatrix} + m_0 c^2 \begin{pmatrix} \mathbb{1}_2 & 0_2 \\ 0_2 & \mathbb{1}_2 \end{pmatrix}\right] \begin{pmatrix} \underline{\Psi}^U(x) \\ \underline{\Psi}^L(x) \end{pmatrix} = \underline{0}$$

$$\left[-\begin{pmatrix} \mathbb{1}_2 & 0_2 \\ 0_2 & \mathbb{1}_2 \end{pmatrix} \imath\hbar \frac{\partial}{\partial t} + c \begin{pmatrix} 0_2 & \boldsymbol{\sigma}\cdot\mathbf{p} \\ \boldsymbol{\sigma}\cdot\mathbf{p} & 0_2 \end{pmatrix} + m_0 c^2 \begin{pmatrix} \mathbb{1}_2 & 0_2 \\ 0_2 & -\mathbb{1}_2 \end{pmatrix}\right] \begin{pmatrix} \underline{\Psi}^U(x) \\ \underline{\Psi}^L(x) \end{pmatrix} = \underline{0} \quad (2.30)$$

where the whole equation has been multiplied first by c and then from the left by $\begin{pmatrix} \mathbb{1}_2 & 0_2 \\ 0_2 & -\mathbb{1}_2 \end{pmatrix}$.

2.1.3.2.2   Dirac Equation for Stationary States

We now focus on stationary states and separate off the time-dependence in the usual way:

$$\underline{\Psi}(x) = \underline{\Psi}(\mathbf{x})\,\Psi(t) = \underline{\Psi}(\mathbf{x})\,e^{-\frac{\imath}{\hbar}Et} \qquad (2.31)$$

which yields, considering that $-\imath\hbar\frac{\partial}{\partial t}\,e^{-\frac{\imath}{\hbar}Et} = -E\,e^{-\frac{\imath}{\hbar}Et}$,

$$\left[c\begin{pmatrix} 0_2 & \boldsymbol{\sigma}\cdot\mathbf{p} \\ \boldsymbol{\sigma}\cdot\mathbf{p} & 0_2 \end{pmatrix} + \begin{pmatrix} m_0 c^2\mathbb{1}_2 & 0_2 \\ 0_2 & -m_0 c^2\mathbb{1}_2 \end{pmatrix}\right]\underline{\Psi}(\mathbf{x}) = E\mathbb{1}_4\,\underline{\Psi}(\mathbf{x})$$

$$(2.32)$$

The Dirac equation has been introduced as a relativistic covariant equation of motion for massive fermions of spin $s = \frac{1}{2}$.

- Relativistic covariant wave equation that treats spatial and time variables on equal footing.

- Correct relativistic energy eigenvalues of the free particle, $E = \pm\sqrt{\mathbf{p}^2 c^2 + m_0^2 c^4}$

- Positive definite probability density, $\rho_D > 0$

2.1.3.2.3   Interpretation of the Dirac Equation

As will be shown, the energy eigenvalues of the free fermion according to Dirac theory are **found** as

$$E = \pm\sqrt{\mathbf{p}^2 c^2 + m_0^2 c^4}. \tag{2.33}$$

The four-dimensional spinor space allows for four solutions, two of which correspond to positive and two of which correspond to negative energy. **The negative energies that only appeared to be a vague possibility earlier are confirmed to be physical reality in Dirac theory**. They are a **mathematical consequence** of the introduction of a first-order Lorentz-covariant differential equation.
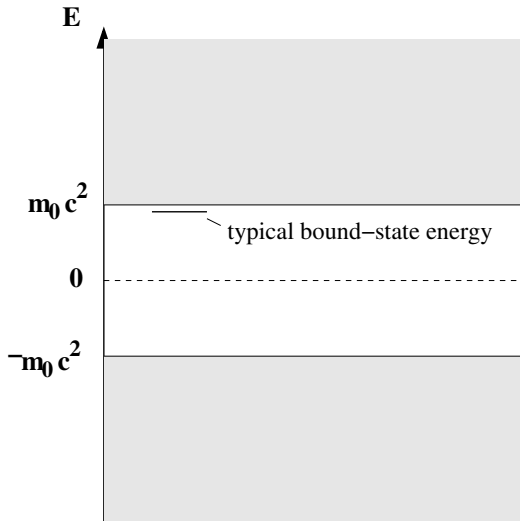
The spectrum of the free fermion according to Dirac theory. As in non-relativistic QM the possible energies are continuous (gray zones), but here for $||\mathbf{p}|| = 0$ we have the rest energy of the particle, $m_0 c^2$, or $-m_0 c^2$ for the branch of negative energies.

The so-called "Dirac gap" (impossible energies for a free particle) is defined as $E_{\text{gap}} = m_0 c^2 - (-m_0 c^2) = 2 m_0 c^2$. Dirac originally formulated the equation for an electron, so let's take $m_0 = m_e$. Then $E_{\text{gap,e}} \approx 1.02$ [MeV]. The total energy (kinetic + potential) of an electron bound in the potential of a proton is -13.61 [eV] (non-relativistic ground state), about 5 orders of magnitude smaller than $E_{\text{gap,e}}$! This is a negative energy for a bound state in non-relativistic theory. In Dirac

theory we have to add the rest energy of the electron, and a resulting bound-state energy[8] is indicated in the above figure (not to scale; to scale it would have to be much closer to the rest energy bar.).

Like in non-relativistic theory of the hydrogen atom there is a discrete spectrum of bound states and a continuous spectrum of scattering states. In relativistic theory these have $E \geq m_0 c^2$, **and** in addition there is a continuous spectrum of negative-energy states! Is this a problem? It is.

For if the "ground state" is no longer the true ground state, i.e., there exist states of lower energy, then the following decay process is quantum-mechanically allowed:

$$p + e \longrightarrow p + e + \gamma \qquad (2.34)$$

The electron could, under emission of a photon of very high energy, transfer to a state of negative energy. This could go on and on, until the hydrogen atom has lost all its energy into radiation. Matter would no longer be stable[9].

Dirac's "Sea" and "Hole theory"

In response to the devastating property of his new equation that predicted that matter should radiate and be unstable, Dirac came up with the following solution: He **postulated** that **all states of negative energy** should be **occupied by the same type of fermions in vacuum**, and this postulate became known as **Dirac sea**. This sea should be perfectly homogeneous and have (among others) the following prop-

---

[8]Of course this result can be obtained by solving the Dirac equation for an electron bound to a proton, but this is a long way to go (much too long for the present course), so I only talk about the result here. In any case, the velocity of an electron in the hydrogen atom is relatively small, so relativistic corrections are small as well. The relativistic total energy – modulo rest energy – is very close to the non-relativistic total energy.

[9]At the time, Dirac's new theory drew fierce criticism from great contemporary physicists. Werner Heisenberg pounded "Dirac's theory is surely the saddest chapter of modern physics!"

erties:

$$m_{\text{sea}} = +\infty$$
$$Q_{\text{sea}} = -\infty \quad \text{(for electrons)}$$
$$E_{\text{sea}} = -\infty$$

Due to its homogeneity no charged particles immersed into it would experience the presence of the sea. And since we ever only measure energy differences in physical processes, never absolute energies, the total infinite energy was not a problem, either. The sea can thus be regarded as a background that has just the right properties to make matter stable.

For it was known that no two identical fermions can occupy the same microstate (Pauli exclusion principle). This means that an electron in any state of positive energy (including its rest energy) could no longer decay to a state of negative energy since all of those were already occupied.

More than this, Dirac postulated as a **direct consequence of the existence of the sea**: If say photons were produced of total energy $E_\gamma > E_{\text{gap,e}}$ the following process could occur:

$$2\gamma \longrightarrow e + \text{“hole''} \tag{2.35}$$

The radiation quanta could "kick" an electron out of the non-observable sea, i.e., excite it to a positive energy and in addition create a hole in the sea[10]. Since the hole corresponds to a "missing electron", its properties must be – according to Dirac –

$$m_{\text{hole}} = m_e$$
$$Q_{\text{hole}} = -Q_e$$
$$E_{\text{hole}} = -E_e$$

---

[10]These radiation quanta could even be produced at very short time scales according to $\Delta E \, \Delta t \geq \frac{\hbar}{2}$, fluctuations of the vacuum that polarize it; a fundamental idea of quantum field theory was born. However, it took another 30 years for this theory to be fully developed and fleshed out.

A hole in the sea has the same inertia as a particle in vacuum, a **missing** charge corresponds to the negative particle charge, and the **missing** energy corresponds to the negative particle energy. Dirac's interpretation was that the hole in the sea had to represent a new particle of equal mass as the electron but of opposite charge (and of positive energy, since it represented the missing of negative energy).

This was one of the boldest and also one of the most spectacular predictions made in science. The particle representing the hole, the "positron", as it was called, was found in cosmic radiation by Anderson and Blackett in 1931, five years after Dirac's prediction of its existence. Since this prediction was not restricted to electrons and any other fermion could replace the electron in the argument, the finding led to the prediction of **antimatter**, i.e., that every type of particle should have a partner with identical mass, but opposite charge. Some particles should therefore be their own antiparticles, like the photon. This concept was later extended to the more general principle of "charge conjugation", $\hat{C}$.

2.1.4   Neutrinos

The basic discoveries in neutrino physics were made between 1930 and 1962. The fundamental observation concerns **nuclear β decay**, where β stands for an electron. At the time, the β decay of an atomic nucleus A into an atomic nucleus B

$$A \longrightarrow B + e \tag{2.36}$$

was understood in terms of the **fundamental process**

$$n \longrightarrow p + e \tag{2.37}$$

i.e., a neutron of nucleus A decays into a proton (a bound proton, so we obtain a new nucleus) and an electron. With the techniques developed

earlier, we are in the position to calculate the energy of the emitted electron, see section 1.7.4. The result of the calculation is, at nuclear level,

$$E_e = \frac{m_A^2 - m_B^2 + m_e^2}{2m_A} c^2 \tag{2.38}$$

Note that this is a fixed value in terms of constants, just like the energy of the emitted muon in section 1.7.4 was. The difference is that here we have two emitted massive particles instead of just one.

Now this theoretical result can be compared to the actual observation in experiment.
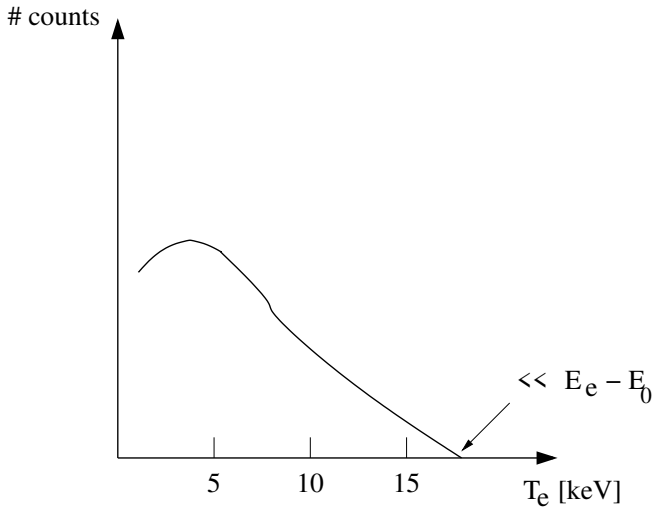
Observed kinetic energy of the emitted electron in the β decay of tritium $^{A=3}_{Z=1}\text{H} \longrightarrow {}^{3}_{2}\text{He} + e$.

A here is the nucleon number and Z is the proton number.

In almost all events the electron's energy is **lower** than the limiting energy calculated *via* Eq. (2.38).

This meant that if conservation of energy should remain valid there is **energy unaccounted for** in the above β decay[11]. Pauli proposed that an unknown additional emitted particle with charge $Q = 0$ should account for the missing energy. Fermi figured out that this new particle must have zero rest mass, and thus the **neutrino** was born. So the

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[11]When confronted with this, Niels Bohr thought that the conservation of energy should be abandoned! However, Bohr was opposed to *many* things at the time, not only to Fermi's neutrino, but also to Dirac's theory, Yukawa's meson, and even Feynman's approach to Quantum Field Theory ...

correct fundamental process can be written as

$$n \longrightarrow p + e + \overline{\nu} \tag{2.39}$$

In fact, it was only later realized that it had to be an *anti*neutrino that is produced here[12].

In the course of these discoveries a general rule was established for particle physics processes:

Crossing symmetry: If a certain reaction is observed then **crossed reactions** are also possible, where **crossed** means that a particle is placed on the other side of the reaction and conjugated into its antiparticle.

For example, a crossed reaction of the fundamental process Eq. (2.39) would be

$$p + \overline{\nu} \longrightarrow n + e^+ \tag{2.40}$$

An electron is here "crossed" into a positron. Cowan and Reynes observed this process in 1955 with solar antineutrinos, and they detected neutron and positron (e^+) formation in the reaction.

In 1953 Konopinski and Mahmoud established the conservation law $L = L'$ of **lepton number**, L, in particle reactions. This can be regarded as an equivalent to charge conservation, $Q = Q'$. A brief survey of lepton numbers:

L	particle type
0	all hadrons
+1	e^-, μ^-, ν
−1	e^+, μ^+, $\overline{\nu}$

Thus, lepton number changes sign when a particle is converted into its antiparticle. In fact, charge conjugation affects all quantum numbers but does not change momentum or energy. Lepton number conservation can be easily confirmed in all of the above processes.

---

[12]Neutrinos and their antiparticles have spin and differ in **helicity**, a concept we might talk about later. Also, conservation laws that were found later on dictated that it had to be an antineutrino of the first generation.

Further confirmations of L conservation followed. Two crossed reactions with respect to Eq. (2.39) are

$$p^+ + \overline{\nu} \longrightarrow n + e^+$$
$$\nu + n \longrightarrow p^+ + e^- \tag{2.41}$$

both of which were observed ($L = L' = -1$ in the first case and $L = L' = +1$ in the second case). On the other hand,

$$\overline{\nu} + n \longrightarrow p^+ + e^- \tag{2.42}$$

which would violate lepton number was never observed.

Unfortunately, this was not the end of the story for the leptons. The decay of the muon according to

$$\mu^- \longrightarrow e^- + \gamma \tag{2.43}$$

is kinematically allowed (the muon is heavier than the electron) and conserves lepton number, but this decay was **never observed**. It was proposed to introduce a conservation law that distinguishes between the three generations, generation 1: electron e^-, generation 2: muon μ^-, generation 3: tau τ^-, i.e., generational lepton numbers L_e, L_μ, L_τ. Then Eq. (2.43) would be forbidden since $L_\mu = 1 \neq L'_\mu = 0$ and $L_e = 0 \neq L'_e = 1$. Using a huge amount of antineutrinos produced in pion decays and testing their reactions with protons, in was in 1962 established that

$$\overline{\nu}_\mu + p^+ \longrightarrow \mu^+ + n \tag{2.44}$$

with $L_\mu = L'_\mu = -1$ and $L_e = L'_e = 0$ takes place whereas

$$\overline{\nu}_\mu + p^+ \longrightarrow e^+ + n \tag{2.45}$$

with $L_\mu = -1 \neq L'_\mu = 0$ and $L_e = 0 \neq L'_e = -1$ **never** does. The true decay channels of the muon and its antiparticle are

$$\mu^- \longrightarrow e^- + \overline{\nu}_e + \nu_\mu \tag{2.46}$$

$$\mu^+ \longrightarrow e^+ + \nu_e + \overline{\nu}_\mu \tag{2.47}$$

where L_μ and L_e are both conserved. We conclude on the first two generations of the lepton family (1962 - 1976) with a summary of their quantum numbers:

lepton	L	L_e	L_μ
e^-	1	1	0
ν_e	1	1	0
μ^-	1	0	1
ν_μ	1	0	1
antilepton	L	L_e	L_μ
e^+	-1	-1	0
$\overline{\nu}_e$	-1	-1	0
μ^+	-1	0	-1
$\overline{\nu}_\mu$	-1	0	-1

2.1.5   Flavor

2.1.5.1   Strangeness and Baryon Number

Between 1947 and 1960 more new hadrons entered the scene, and their observed behavior allowed for an extension of the conservation laws known thus far. The heavy meson K^0 (composed of a linear combination of a strange s and an antidown $\overline{d}$ quark and vice versa) and the baryon $\Lambda(uds)$ **decay under weak interaction** as follows:

$$K^0 \longrightarrow \pi^+ + \pi^- \tag{2.48}$$

$$\Lambda \longrightarrow p^+ + \pi^- \tag{2.49}$$

The K^0 has meson number $+1$, just like the π^+. So the π^- has meson number -1 since it is the antiparticle of the π^+. This means that **meson number is generally not conserved**. The same conclusion can be drawn from the decay of the Λ. On the left-hand side meson number is 0, but on the right-hand side meson number is -1. These new relatively heavy mesons were called "strange" particles (the quark decomposition became known only later!), mainly because their creation – driven by the strong interaction – is a relatively fast process, but their decay – driven by the weak interaction – is relatively slow; the difference is orders of magnitude.

On the other hand, **baryon number, B, is conserved**[13]. The conservation of B can also be verified on Eq. (2.39). Some important baryon numbers:

B	particle type
0	all leptons, all mesons
$+1$	p^+, n, Λ
-1	p^-, $\overline{n}$

The antiproton $p^- \equiv \overline{p}$ was first produced in the following inelastic collision:

$$p^+ + p^+ \longrightarrow p^+ + p^+ + p^+ + p^- \tag{2.50}$$

Note that this is a "sticky" relativistic collision. Conservation of Q (total charge), B, L_e and L_μ are easily verified.

Ongoing investigations and results and the early days of the quark model affirmed that a new quantum number could be introduced, called "strangeness" (S), that was **conserved** in processes driven by the **strong interaction**, but **not conserved** in processes driven by the

---

[13]B is **almost** always conserved in particle processes. It took another while to find a rare exception which is connected to charge-parity (CP) violation (the K^0 meson decays in 0.2% of events under violation of CP.). In order to formalize this we need to understand how to write the operators $\hat{C}$ (charge conjugation) and $\hat{P}$ (space inversion) in the framework of Dirac theory. This is the subject of more advanced chapters.

weak interaction. Examples:

$$\pi^-(d\bar{u}) + p^+(uud) \longrightarrow K^+(u\bar{s}) + \Sigma^-(dds) \qquad (2.51)$$

This is a strong-interaction process. We observe $B = B' = +1$, $Q = Q' = 0$, and $S = 0 + 0 = S' = 1 + (-1)$. The strange quark, s, was given $S = -1$ and its antipartner $\bar{s}$ has $S = +1$. So strangeness is conserved.

Now consider again the weak decay in Eq. (2.49). $S = -1 \neq S' = 0$ and strangeness is not conserved.

2.1.5.2   The Eightfold Way

The situation having become ever more chaotic, Gell-Mann and Ne'eman in the period of 1961 - 1964 invented an ordering scheme called the "**Eightfold Way**" that not only helped understand particle phenomenology but that also made successful predictions of so far unknown particles!
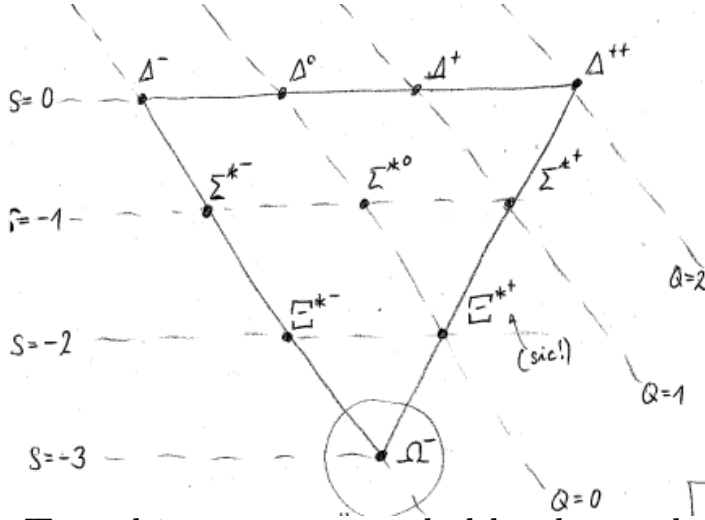
Gell-Mann and Ne'eman realized that the **eight lightest baryons** could be organized **into an octet**, according to charge and strangeness.
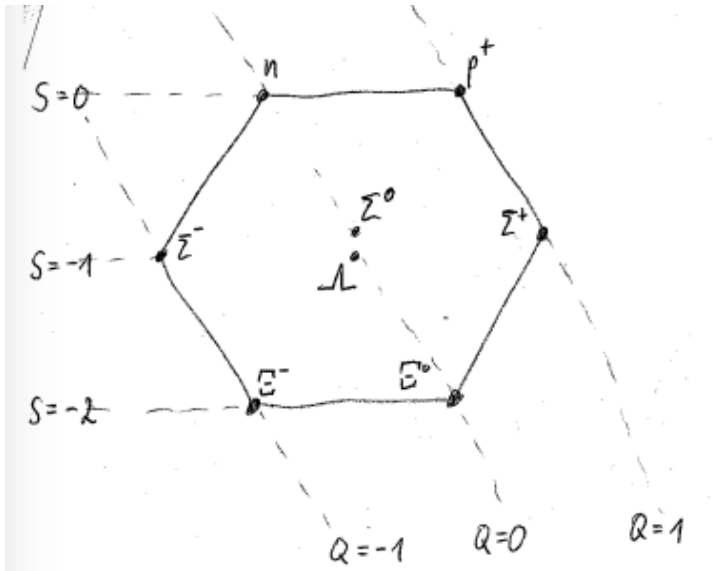
The octet of light baryons. The quark decomposition for the "new" baryons is $\Theta^-(dss)$, $\Theta^0(uss)$, $\Sigma^0(uds)$, $\Sigma^+(uus)$.

Likewise, the ten next heavier baryons form a decuplet where iso-

axes of charge and strangeness are the same as in the octet scheme.

The decuplet of heavier baryons. The quark decomposition of these baryons is $\Delta^-(ddd)$, $\Delta^0(udd)$, $\Delta^+(uud)$, $\Delta^{++}(uuu)$, $\Sigma^{*-}(dds)$, $\Sigma^{*0}(uds)$, $\Sigma^{*+}(uus)$, $\Theta^{*-}(dss)$, $\Theta^{*0}(uss)$, $\Omega^-(sss)$.

Two things are remarkable about the baryon decuplet diagram. The first is that the Ω^- forming the lower corner was not known at the time of its making. It was a prediction that was shortly afterwards confirmed!

Second, some of these baryons have the same quark decomposition as the lighter baryons, for example $\Theta^{*-}(dss)$ and $\Theta^-(dss)$. The difference is that in the Θ^{*-} the three quarks are **confined** in an **excited state** which is denoted by the asterisk ($*$). So we would expect them, according to Eq. (1.138), to have **different rest mass**. Indeed,

$$m_{\Theta^{*-}} = 1533 \left[\frac{\text{MeV}}{c^2}\right]; \quad S = 3/2$$

$$m_{\Theta^-} = 1321 \left[\frac{\text{MeV}}{c^2}\right]; \quad S = 1/2$$

In addition, they have different spin quantum numbers S. Similarly, the proton $p^+(uud)$ has rest mass $m_{p+} = 938 \left[\frac{\text{MeV}}{c^2}\right]$ whereas the excited $\Delta^+(uud)$ has $m_{\Delta+} = 1232 \left[\frac{\text{MeV}}{c^2}\right]$.

This raises an important question: When do we consider an excited-state particle as a **different particle**? We might compare the situation with atomic physics and ask whether an excited hydrogen atom, H^*, should not be regarded as a **different particle**, too, compared to

H, since it has higher rest mass than H. However, typical atomic excitation energies are on the order of [eV], and the rest energy of the proton is $E_{0,p^+} = 938$ [MeV]. This is a difference of about 9 orders of magnitude! In the above baryons, on the other hand, rest energy and excitation energy are in the same order of magnitude, $O(E_0) \approx O(E^*)$. This is why we here speak of a different particle whereas for excited atoms we do not.
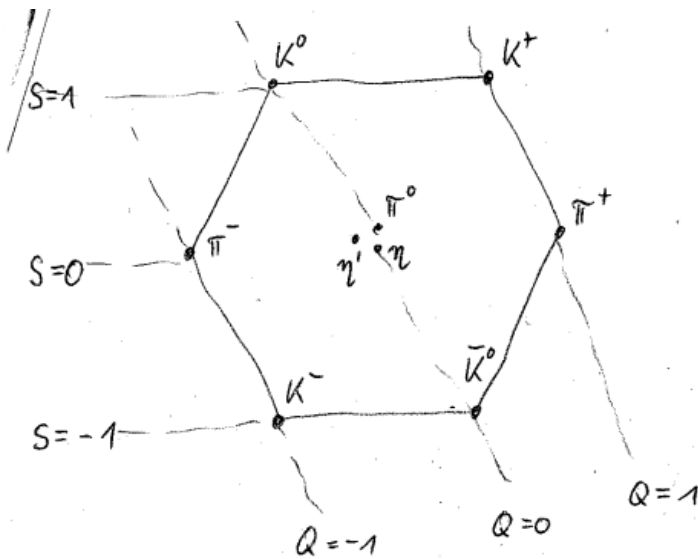
Finally, the lightest mesons are organized into a **meson nonet**.   The η particles are linear combinations of $(u\bar{u})$, $(d\bar{d})$, and $(s\bar{s})$ states, the π^0 of $(u\bar{u})$ and $(d\bar{d})$ states.

2.1.5.3   Quark model and Eightfold Way

In 1964 Murray Gell-Mann and George Zweig introduced the solution to the question as of how the above Eightfold-Way diagrams **emerge** from a deeper, underlying structure.
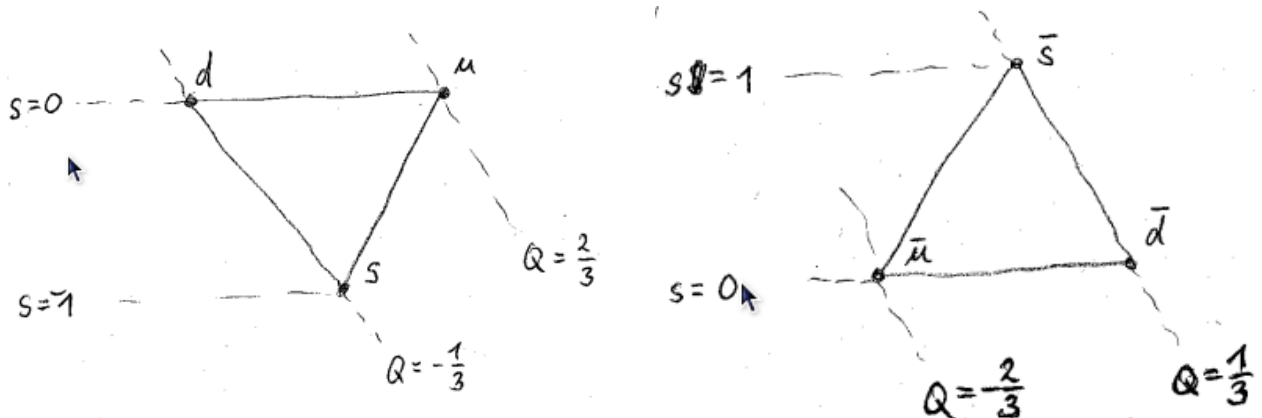
Three quarks and their **anti**partners, organized as above into triangular diagrams, can account for the observed bound states of baryons and mesons[14]. In essence, the quark model conjectures that

1. All (anti)baryons are composed of 3 (anti)quarks.

2. All mesons are composed of one quark and one antiquark.

Quarks are **confined**[15] into bound states and are never observed individually.

2.1.5.4 Problems With the Quark model – and Some Solutions

The first problem with the quark model is that they are not detectable individually. However, scattering experiments on baryons show (similar to the early scattering experiments by Rutherford on atoms) that there are three localized mass "concentrations" in a baryon, pointing to the quark decomposition.

The second problem becomes obvious when thinking about the existence of, for example, the $\Delta^{++}(uuu)$ particle (see the baryon decuplet diagram). It has total spin $S = \frac{3}{2}$ and is composed of three identical particles, u quarks, which each have $s = \frac{1}{2}$. This means that at least two of these quarks must have identical quantum numbers, $m_s(i) = \frac{1}{2}$, which, however, is forbidden by the Pauli exclusion principle!

In 1964 Greenberg proposed a solution to this problem. First of all, particles had been given the following **"flavor"** quantum numbers:

---

[14]There is quite a bit more to be said here, for example why the corners of the baryon octet diagram are "missing" compared to the baryon decuplet diagram. For answering this we would have to analyze the irreducible representations of color SU(3) ⊗ spin SU(2).

[15]Just like us, these days ...

Quantum number	Name	Example
U	"upness"	u, $U = +1$
D	"downness"	d, $D = +1$, $U = 0$
S	"strangeness"	s, $S = -1$, $U = 0$, etc.
C	"charm"	c, $C = +1$, $U = 0$, etc.
B	"bottomness"	b, $B = -1$, $U = 0$, etc.
T	"topness"	t, $T = +1$, $U = 0$, etc.

which add up in composed particles, so $U = 0$ for the π^0 meson. In addition to flavor, Greenberg hypothesized that quarks come in "colors", i.e., "red", "green", and "blue". These are additional properties that are chosen in analogy with the property of light being colorless when all three wavelengths are mixed. Likewise, all occurring particles are "colorless". This solves the problem: If the three u quarks in the $\Delta^{++}(uuu)$ have three different colors, then no two of them can have the same total set of quantum numbers. For consistency, the mesons are colorless, too. For example, the π^0 ($u\overline{u}$, $d\overline{d}$) is colorless when the quark-antiquark pair has the same color. "red" + "antired" $\longrightarrow$ colorless, likewise for blue and green pairs.

This may sound very daring, but Greenberg's idea of course makes observable predictions. For example, a corrolary is that bound states consisting of two quarks (or two antiquarks) should not exist, because they would not be colorless. Likewise for bound states consisting of four quarks or four antiquarks. Indeed, no such particles have been observed[16].

The introduction of flavor and color made further predictions that were later on spectacularly confirmed, for example with the discovery of the Ψ ($c\overline{c}$) and D^+ ($c\overline{d}$) particles in 1974 and 1976.

---

[16]In a very recent discovery, however, tetraquarks, composed of two quark-antiquark pairs – for example the bound system ($c\overline{c}s\overline{s}$) – have been detected at CERN (arXiv:2103.01803 [hep-ex]).

2.1.5.5 Standard Model

I want to conclude this chapter with a summary of the known particles composing the Standard Model of elementary particles.

Leptons		Quarks		Force mediators	Higgs
e^-	e^+	d	$\overline{d}$	γ (photon, EM)	H
ν_e	$\overline{\nu}_e$	u	$\overline{u}$	$W^\pm$, Z^0 (vector bosons, weak)	
μ^-	μ^+	s	$\overline{s}$	8 gluons (strong)	
ν_μ	$\overline{\nu}_\mu$	c	$\overline{c}$		
τ^-	τ^+	b	$\overline{b}$		
ν_τ	$\overline{\nu}_\tau$	t	$\overline{t}$		

There are 37 particles altogether. The 8 gluons are classified according to their color transmission ($r\overline{r}, r\overline{g}, r\overline{b}, g\overline{g}, g\overline{b}, g\overline{r}, b\overline{b}, b\overline{r}, b\overline{g}$). The dimension of this set is 9, and they span two irreducible representations of color $SU(3)$, $\dim(\Gamma^1) = 1$ and $\dim(\Gamma^2) = 8$. The totally symmetric Γ^1 representation is not observed in nature, leaving 8 linear combinations of color transmissions.