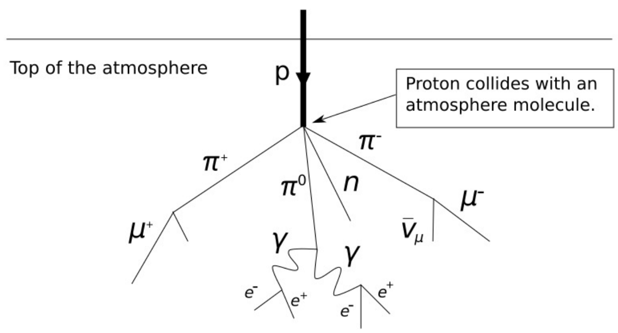
Chapter 2

(A Brief) Introduction to Elementary Particles

2.1 Standard Model Phenomenology

2.1.1 Mesons



2.1.2 Antimatter - Dirac Equation

2.1.2.1 Klein-Gordon Theory

2.1.2.1.1 Derivation of the Klein-Gordon Equation

Starting point is the time-dependent Schrödinger equation (SEQ):

$$\imath\hbar \frac{\partial}{\partial t} \Psi(\vec{x}, t) = \hat{H}\, \Psi(\vec{x}, t) \tag{2.1}$$

is a differential equation first order in time and second order in space and, therefore, cannot be Lorentz covariant. To understand this consider the one-dimensional SEQ, slightly rewritten:

$$\hbar \left(\imath \frac{\partial}{\partial t} + \frac{\hbar}{2m} \frac{\partial^2}{\partial x^2} \right) \Psi(\vec{x}, t) = 0 \tag{2.2}$$

We know that $\partial^\mu \partial_\mu$ is a Lorentz scalar, so $a \frac{\partial}{\partial t} + b \frac{\partial^2}{\partial x^2}$ with constant a, b cannot be one.

As a first step, Klein and Gordon took the time derivative of Eq. (2.1)

$$\imath\hbar \frac{\partial^2}{\partial t^2} \Psi(\vec{x}, t) = \hat{H} \frac{\partial}{\partial t} \Psi(\vec{x}, t). \tag{2.3}$$

Note that we are working with a time-independent Hamiltonian operator in the Schrödinger picture. After introduction of Eq. (2.1) in the form $\frac{\partial}{\partial t} \Psi(\vec{x}, t) = \frac{1}{\imath\hbar} \hat{H} \Psi(\vec{x}, t)$ into the right-hand side of Eq. (2.3) we get

$$\imath\hbar \frac{\partial^2}{\partial t^2} \Psi(\vec{x}, t) = \frac{1}{\imath\hbar} \hat{H}^2 \Psi(\vec{x}, t). \tag{2.4}$$

This is so far not a deviation from non-relativistic quantum physics, and Eq. (2.4) is still not homogeneous in time and space derivatives when considering the non-relativistic energy-momentum relation[1].

Now, Klein and Gordon argued that Einstein's energy of a particle in relativistic mechanics is $E = \pm\sqrt{\vec{p}^{\,2} c^2 + m_0^2 c^4}$, and so we replace the Hamiltonian in Eq. (2.4) by this expression, introducing position space operators for momentum $\hat{\vec{p}}^{\,2} = -\hbar^2 \vec{\nabla}^2$

$$\begin{aligned} \left(\hbar^2 \frac{\partial^2}{\partial t^2} - \hbar^2 c^2 \vec{\nabla}^2 + m_0^2 c^4 \right) \Psi(\vec{x}, t) &= 0 \\ \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + \frac{m_0^2 c^2}{\hbar^2} \right) \Psi(\vec{x}, t) &= 0, \end{aligned} \tag{2.5}$$

which is equivalent to the procedure in non-relativistic quantum mechanics where the non-relativistic energy momentum relation is used at this point.

[1]One might ponder over using the square root directly in Eq. (2.1), but this results in operator roots and a mathematically quite complicated equation.

In Minkowski space with a metric tensor

$$\{g_{\mu\nu}\} = \{g^{\mu\nu}\} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.6}$$

and real-valued coordinates, so $\{x_\mu\} := \begin{pmatrix} x_0 = ct \\ \mathbf{x} \end{pmatrix}$, the time derivative can be written like

$$\frac{1}{c}\frac{\partial}{\partial t} = \frac{1}{c}\frac{\partial x_0}{\partial t}\frac{\partial}{\partial x_0} = \frac{1}{c}c\frac{\partial}{\partial x_0} = \frac{\partial}{\partial x_0} =: \partial^0 \tag{2.7}$$

and thus Eq. (2.5) becomes

$$\begin{aligned} \left(\frac{\partial^2}{\partial x_0^2} - \sum_k \frac{\partial^2}{\partial x_k^2} + \frac{m_0^2 c^2}{\hbar^2}\right)\Psi(\vec{x}, t) &= 0 \\ \left(\partial^\mu \partial_\mu + \frac{m_0^2 c^2}{\hbar^2}\right)\Psi(x) &= 0 \end{aligned} \tag{2.8}$$

the **Klein-Gordon equation** where in the last step the position four-vector x replaces the coordinates $\vec{x}, t$ in the argument of the wave function. The operator $\Box^2 = \partial^\mu \partial_\mu$ is called the **d'Alembertien**.

It can be shown that the KG equation is indeed Lorentz covariant, *i.e.*, it retains its form under Lorentz transformations. This becomes manifest in the structure of the KG operator which is obviously a Lorentz scalar.

2.1.2.1.2 Problems with Klein-Gordon Theory

It was, however, quickly realized that Klein and Gordon could not claim victory in having solved the problem of formulating the fundamental equation of relativistic quantum mechanics. The central problem with Eq. (2.8) is as follows.

In analogy to Schrödinger theory the conservation of the probability density is assured by it satisfying a continuity equation with an appropriate probability current density. For Klein-Gordon theory, this current density is

$$j^{\mu} := \frac{\imath\hbar}{2m_0}\left[\psi^{*}\left(\partial^{\mu}\psi\right) - \left(\partial^{\mu}\psi\right)^{*}\,\psi\right] \tag{2.9}$$

which shall be demonstrated.

Proof.

$$\partial_{\mu}j^{\mu} = \frac{\imath\hbar}{2m_0}\left[\left(\partial_{\mu}\psi^{*}\right)\left(\partial^{\mu}\psi\right) + \psi^{*}\left(\partial_{\mu}\partial^{\mu}\psi\right) - \left(\partial_{\mu}\partial^{\mu}\psi\right)^{*}\,\psi - \left(\partial^{\mu}\psi\right)^{*}\left(\partial_{\mu}\psi\right)\right] \tag{2.10}$$

The last term in Eq. (2.10) can be rewritten as

$$\begin{aligned} -\left(\partial^{\mu}\psi\right)^{*}\left(\partial_{\mu}\psi\right) &= -\left(\partial_{\nu}g^{\nu\mu}\psi\right)\left(\partial^{\kappa}g_{\kappa\mu}\,\psi\right) = -\left(\partial_{\nu}\psi\right)g^{\nu\mu}g_{\mu\kappa}\left(\partial^{\kappa}\psi\right) &(2.11)\\ &= -\left(\partial_{\nu}\psi\right)\delta^{\nu}_{\kappa}\left(\partial^{\kappa}\psi\right) = -\left(\partial_{\nu}\psi^{*}\right)\left(\partial^{\nu}\psi\right) &(2.12)\end{aligned}$$

and so cancels with the first term. Using the KG equation Eq. (2.8) in the second and third terms of Eq. (2.10) results in

$$\begin{aligned}\partial_{\mu}j^{\mu} &= \frac{\imath\hbar}{2m_0}\left[\psi^{*}\left(-\frac{m_0^2c^2}{\hbar^2}\,\psi\right) - \left(-\frac{m_0^2c^2}{\hbar^2}\,\psi^{*}\right)\psi\right]\\ &= \frac{\imath\hbar}{2m_0}\left[-\frac{m_0^2c^2}{\hbar^2} + \frac{m_0^2c^2}{\hbar^2}\right]\psi^{*}\psi = 0 \end{aligned} \tag{2.13}$$

which shows that the chosen probability current density in Eq. (2.9) indeed leads to a consistent KG theory. Since the probability current density four vector has the form given in Eq. (1.98) the KG probability density is

$$\begin{aligned}\rho_{\mathrm{KG}} &= \frac{\imath\hbar}{2m_0}\left[\psi^{*}\left(\partial^{0}\psi\right) - \left(\partial^{0}\psi\right)^{*}\,\psi\right]\\ &= \frac{\imath\hbar}{2m_0}\left[\psi^{*}\left(\frac{1}{c}\frac{\partial}{\partial t}\psi\right) - \left(\frac{1}{c}\frac{\partial}{\partial t}\psi\right)^{*}\,\psi\right]. \end{aligned} \tag{2.14}$$

The solutions of the KG equation are of plane-wave type and can be written as

$$\psi_{\mathrm{KG}}(x) = Ae^{\imath(\omega t - \mathbf{k}\cdot\mathbf{x})} + Be^{-\imath(\omega t - \mathbf{k}\cdot\mathbf{x})} \tag{2.15}$$

which when inserted into the KG equation yields the energy-momentum relation of relativistic theory. Using Eq. (2.15) in the representation of the KG density, Eq. (2.14) results in

$$\rho_{\mathrm{KG}} \propto -|A|^2 + |B|^2 \tag{2.16}$$

which means that the KG probability density could, depending on specific initial conditions which determine the coefficients A and B, become negative. This, however, is in stark contradiction with the fundamentals of quantum mechanics where probability density is always positive definite!

Historically, the KG equation was, therefore, regarded as a complete failure and discarded.

2.1.2.2 Dirac Theory

2.1.2.2.1 Derivation of the Dirac Equation

Dirac realized that the essential problem with the KG equation was the fact that it was a second-order differential equation. Thus, his idea was to formulate a Lorentz covariant **first-order** differential equation that treated space and time coordinates on an equal footing.

He started out by introducing a set of general parameters $\{\gamma^\mu\}$, the properties of which had to be determined, to see if the square root of the d'Alembertien could be developed in such a way that it takes on the form of a Lorentz scalar. Rewriting it as

$$\begin{aligned}\partial_0\partial_0 - \sum_k \partial_k\partial_k \;=\; & \gamma^0\gamma^0\,\partial_0\partial_0 + \gamma^1\gamma^1\,\partial_1\partial_1 + \gamma^2\gamma^2\,\partial_2\partial_2 + \gamma^3\gamma^3\,\partial_3\partial_3 \\ & + \left\{\gamma^0,\gamma^1\right\}\,\partial_0\partial_1 + \left\{\gamma^0,\gamma^2\right\}\,\partial_0\partial_2 + \left\{\gamma^0,\gamma^3\right\}\,\partial_0\partial_3 \\ & + \left\{\gamma^1,\gamma^2\right\}\,\partial_1\partial_2 + \left\{\gamma^1,\gamma^3\right\}\,\partial_1\partial_3 + \left\{\gamma^2,\gamma^3\right\}\,\partial_2\partial_3\end{aligned}$$

where $\{,\}$ is the anti-commutator would be correct if

$$\{\gamma^\mu, \gamma^\nu\} = 0 \qquad \forall \mu \neq \nu \tag{2.17}$$
$$\{\gamma^0, \gamma^0\} = 2 \tag{2.18}$$
$$\{\gamma^k, \gamma^k\} = -2 \qquad \forall k \in \{1 \ldots 3\} \tag{2.19}$$

For example, since $\{\gamma^0, \gamma^0\} = 2\gamma^0, \gamma^0 \Rightarrow \gamma^0, \gamma^0 = 1$ is in accord with the above decomposition. The quantities γ are today known as **Clifford numbers** and the equations (2.18), (2.19), (2.19) as the conditions for a **Clifford algebra**.

Let's take a look at the first condition, Eq. (2.18). If $\gamma^\kappa \in \mathbb{C}$ is just a complex scalar, then $\{\gamma^\mu, \gamma^\nu\} = 2\gamma^\mu\gamma^\nu = 0$, since scalars commute. This can only be solved by setting $\gamma^\mu = 0 \;\; \forall \mu$ (or likewise $\gamma^\nu = 0 \;\; \forall \nu$) which does not lead to a valid set of parameters. Dirac concluded that the set $\{\gamma^\mu_{n\times n}\}$ had to be $n \times n$ matrices in order to arrive at anticommutators that yield zero!

Dirac's first try was to use the set $\{\mathbf{1}, \boldsymbol{\sigma}\}$ which form a basis for all complex 2×2 matrices. However, neither this choice leads to a valid decomposition of the d'Alembertien[2].

Dirac found the simplest set to be

$$\gamma^0 := \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} \qquad \gamma^k := \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma}^k \\ -\boldsymbol{\sigma}^k & \mathbf{0} \end{pmatrix} \tag{2.20}$$

where the spin-Pauli matrices have been introduced[3]. Now it is straightforward to rewrite the d'Alembertien as

$$\left(\partial_0\partial_0 - \sum_k \partial_k\partial_k\right) \mathbb{1}_4 = \gamma^\mu \gamma^\nu \partial_\mu\partial_\nu = \gamma^\mu\partial_\mu \gamma^\nu\partial_\nu = (\gamma^\mu\partial_\mu)^2 \tag{2.21}$$

[2] It can be shown in a general manner that only matrices with dimension multiples of 4 are possible solutions. So, for example, dimension 8 matrices can be constructed, but the resulting theory is identical to the dimension 4 theory in physical content.

[3] This set of matrices is called the "standard representation" of Dirac matrices. Other representations related through unitary transformations of the standard matrices are possible as well.

and so

$$\sqrt{\Box^2}\,\mathbb{1}_4 = \gamma^\mu \partial_\mu \tag{2.22}$$

Note that we here have obtained $\Box^2$ as the product of two (formally) Lorentz scalars. This is **not** the same thing as the Minkowski space scalar product between the four-vectors $\partial_\mu \partial^\mu$!

A direct consequence of this astonishing finding is that the wavefunction onto which such an operator acts cannot be a scalar function. It has to be a four-component vector-like quantity.

Dirac now formulated his famous equation for a free particle of rest mass m_0 as

$$(-\imath\hbar\,\gamma^\mu\,\partial_\mu + m_0 c\,\mathbb{1}_4)\;\underline{\Psi}(x) = \underline{0} \tag{2.23}$$

which is a matrix equation where the wavefunction becomes a **four-component spinor** that depends on the position four-vector x (which contains the time coordinate).

Using the standard representation the Dirac equation can be written out more explicitly as

$$\begin{aligned} \left\{-\imath\hbar\left[\gamma^0\partial_0 + \gamma^1\partial_1 + \gamma^2\partial_2 + \gamma^3\partial_3\right] + m_0 c\mathbb{1}_4\right\}\underline{\Psi}(x) &= \underline{0} \\ \left\{-\imath\hbar\left[\begin{pmatrix} \mathbb{1}_2 & 0_2 \\ 0_2 & -\mathbb{1}_2 \end{pmatrix}\frac{\partial}{\partial x_0} + \sum_{k=1}^{3}\begin{pmatrix} 0_2 & \boldsymbol{\sigma}_k \\ -\boldsymbol{\sigma}_k & 0_2 \end{pmatrix}\frac{\partial}{\partial x_k}\right] + m_0 c\begin{pmatrix} \mathbb{1}_2 & 0_2 \\ 0_2 & \mathbb{1}_2 \end{pmatrix}\right\}\begin{pmatrix} \underline{\Psi}^U(x) \\ \underline{\Psi}^L(x) \end{pmatrix} &= \underline{0}. \end{aligned} \tag{2.24}$$

The second line follows from the realization that the γ matrices have a 2×2 block structure, and so the entire equation can be written in this so-called “bi-spinor” form. The associated 2-spinors are called “upper” ($\underline{\Psi}^U(x)$) and “lower” ($\underline{\Psi}^L(x)$) 2-spinors. Their significance will become clear when solutions of the free-particle Dirac equation are investigated.

Now, since in position-space representation $\mathbf{p} = -\imath\hbar\,\boldsymbol{\nabla}$ it is convenient to rewrite the term involving the spin-Pauli matrices using the scalar product between the 3-vector of the Pauli matrices and the 3-

vector of momentum, $\boldsymbol{\sigma} \cdot \mathbf{p} = \boldsymbol{\sigma}_x \hat{p}_x + \boldsymbol{\sigma}_y \hat{p}_y + \boldsymbol{\sigma}_z \hat{p}_z$,[4] and the Dirac equation becomes

$$\left[-\begin{pmatrix} \mathbb{1}_2 & 0_2 \\ 0_2 & -\mathbb{1}_2 \end{pmatrix} \imath\hbar\frac{\partial}{\partial t} + c\begin{pmatrix} 0_2 & \boldsymbol{\sigma}\cdot\mathbf{p} \\ -\boldsymbol{\sigma}\cdot\mathbf{p} & 0_2 \end{pmatrix} + m_0c^2\begin{pmatrix} \mathbb{1}_2 & 0_2 \\ 0_2 & \mathbb{1}_2 \end{pmatrix}\right]\begin{pmatrix} \underline{\Psi}^U(x) \\ \underline{\Psi}^L(x) \end{pmatrix} = \underline{0}$$

$$\left[-\begin{pmatrix} \mathbb{1}_2 & 0_2 \\ 0_2 & \mathbb{1}_2 \end{pmatrix} \imath\hbar\frac{\partial}{\partial t} + c\begin{pmatrix} 0_2 & \boldsymbol{\sigma}\cdot\mathbf{p} \\ \boldsymbol{\sigma}\cdot\mathbf{p} & 0_2 \end{pmatrix} + m_0c^2\begin{pmatrix} \mathbb{1}_2 & 0_2 \\ 0_2 & -\mathbb{1}_2 \end{pmatrix}\right]\begin{pmatrix} \underline{\Psi}^U(x) \\ \underline{\Psi}^L(x) \end{pmatrix} = \underline{0} \quad (2.25)$$

where the whole equation has been multiplied first by c and then from the left by $\begin{pmatrix} \mathbb{1}_2 & 0_2 \\ 0_2 & -\mathbb{1}_2 \end{pmatrix}$.

2.1.2.2.2 Dirac Equation for Stationary States

We now focus on stationary states and separate off the time-dependence in the usual way:

$$\underline{\Psi}(x) = \underline{\Psi}(\mathbf{x})\,\Psi(t) = \underline{\Psi}(\mathbf{x})\,e^{-\frac{\imath}{\hbar}Et} \quad (2.26)$$

which yields, considering that $-\imath\hbar\frac{\partial}{\partial t}\,e^{-\frac{\imath}{\hbar}Et} = -E\,e^{-\frac{\imath}{\hbar}Et}$,

$$\left[c\begin{pmatrix} 0_2 & \boldsymbol{\sigma}\cdot\mathbf{p} \\ \boldsymbol{\sigma}\cdot\mathbf{p} & 0_2 \end{pmatrix} + \begin{pmatrix} m_0c^2\mathbb{1}_2 & 0_2 \\ 0_2 & -m_0c^2\mathbb{1}_2 \end{pmatrix}\right]\underline{\Psi}(\mathbf{x}) = E\mathbb{1}_4\,\underline{\Psi}(\mathbf{x}) \quad (2.27)$$

The Dirac equation has been introduced as a relativistic covariant equation of motion for massive fermions of spin $s = \frac{1}{2}$.

- Relativistic covariant wave equation that treats spatial and time variables on equal footing.
- Correct relativistic energy eigenvalues of the free particle, $E = \pm\sqrt{\mathbf{p}^2c^2 + m_0^2c^4}$
- Positive definite probability density, $\rho_D > 0$

[4]Note, e.g., that the product of a Pauli matrix with a scalar momentum operator is well defined: $\boldsymbol{\sigma}_x\hat{p}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\hat{p}_x = \begin{pmatrix} 0 & \hat{p}_x \\ \hat{p}_x & 0 \end{pmatrix}$.