1.5 Relativistic Mass and Linear Momentum

We have seen in the preceding section that the equation of motion of classical mechanics has been generalized to Minkowski space and is formulated in terms of four-vectors. In the following subsections we want to investigate the consequences of this generalization. We begin from a space-like component of the relativistic equation of motion (1.103) and rewrite it:

$$\begin{aligned} m_0\, b^k &= K^k \\ m_0 \frac{du^k}{d\tau} &= \gamma\, F_k \\ m_0 \frac{du^k}{dt} &= F_k \\ \frac{d\,(m_0\,\gamma\, v_k)}{dt} &= F_k \end{aligned}$$

where the definition of four-acceleration (1.99), the proper time differential (1.93), and the obtained expression for four-velocity (1.97) have been used. Form equivalence with Eq. (1.106) and dimensional analysis suggest to define **relativistic linear momentum** as

$$p^k \equiv m_0\,\gamma\, v_k. \tag{1.129}$$

So we have established the space-like components of the linear-momentum four-vector. Before completing the four-vector it is instructive to inspect Eq. (1.129) more closely.

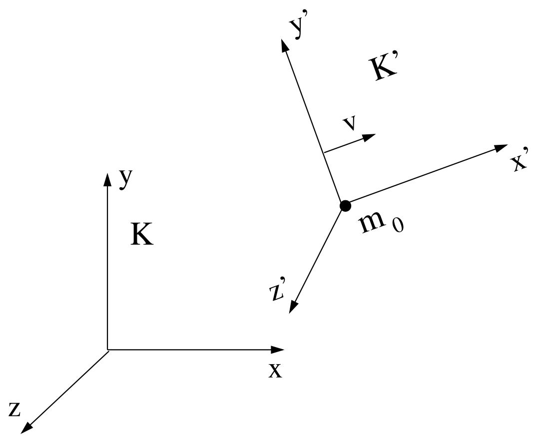
1.5.1 Relativistic Mass

In the framework of classical Newtonian mechanics $\mathbf{p} = m\,\mathbf{v}$ where m is the inertial mass of a given particle or body. Since in Eq. (1.129) v_k is the velocity of the particle in frame K and p^k the associated momentum, the implication is that

$$m := \gamma\, m_0 \tag{1.130}$$

should be regarded as the particle's **relativistic inertial mass** in frame K, instead of simply the the rest mass m_0 of the particle. This is a profound difference and means that, since $\gamma = \gamma(v)$ is a function of the velocity of the particle in frame K, so is its mass. The finding is illustrated in Fig. (1.13). Note that rest mass is a Lorentz scalar, i.e.,

Figure 1.13:



A massive particle with rest mass m_0 in frame K' moves with velocity $\mathbf{v}$ relative to frame K. Its relativistic mass in coordinates of frame K is $\gamma(v)\, m_0$.

it does not depend on any state of movement. However, in coordinates of frame K, the particle "behaves" as if it had an increased mass, its relativistic or dynamic mass (or observed mass)[29]. It can be anticipated that also the expression for the energy of the particle in K should differ from that in frame K', but that is yet to be substantiated.

1.5.2 Relativistic Linear Momentum

With these conclusions in mind, the relativistic generalization of linear momentum is straightforward. Since the space-like components are proportional to the dynamic particle mass and its velocity in frame K, the time-like component results by analogy and using the expression for the velocity four-vector in Eq. (1.97):

$$p^0 = m_0\, \gamma\, c = m_0\, u^0 \tag{1.131}$$

[29]The concept of relativistic mass is not a fundamental requirement. In fact, many authors argue against its introduction since it is sufficient to consider relativistic momentum modified by the γ factor. However, dynamic mass is a useful way of thinking about various situations for instance in the physics of an atom.

Summary for the linear momentum four-vector:

$$\{p^\mu\} = \left\{ \begin{array}{c} p^0 \\ p^k \end{array} \right\} = \left\{ \begin{array}{c} m_0\, u^0 \\ m_0\, u^k \end{array} \right\} \equiv \left\{ \begin{array}{c} mc \\ mv_k \end{array} \right\} \qquad \forall k \in \{1, \ldots, 3\} \quad (1.132)$$

As a check for consistency, we take the proper-time derivative of linear momentum,

$$\frac{d}{d\tau}\, p^\mu = m_0\, \frac{d}{d\tau}\, u^\mu = m_0\, b^\mu = K^\mu \qquad (1.133)$$

where the relativistic equation of motion (1.103) has been used. We thus obtain a law analogous in form to its non-relativistic counterpart. We can also verify Lorentz covariance on this last equation. Since $\frac{d}{d\tau}$ is a Lorentz scalar, the l.h.s. (left-hand side) transforms like a contravariant four vector, and so does the r.h.s.

1.6 Relativistic Energy

We now start from the time-like component of the relativistic fundamental law of dynamics (1.103) and obtain

$$\begin{aligned} m_0\, b^0 &= K^0 \\ m_0\, \frac{du^0}{d\tau} &= \frac{\gamma}{c}\, \mathbf{F} \cdot \mathbf{v} \\ m_0\, \gamma\, \frac{du^0}{dt} &= \frac{\gamma}{c}\, \mathbf{F} \cdot \mathbf{v} \\ \frac{d}{dt}\left(m_0\, \gamma(v)\, c^2\right) &= \mathbf{F} \cdot \mathbf{v} \qquad (1.134) \end{aligned}$$

where the expression for the proper time differential (1.93) and the time-like component of four-velocity (1.97) have been used.

Integrating Eq. (1.134) over time results in

$$\int \frac{d}{dt}\left(m_0\, \gamma(v)\, c^2\right) dt = \int \mathbf{F} \cdot \mathbf{v} dt \qquad (1.135)$$

$$m_0\, \gamma(v)\, c^2 = \int \mathbf{F} \cdot \mathbf{dx} \qquad (1.136)$$

using $\mathbf{v}dt = \mathbf{dx}$.

Now, since work is $W = \int \mathbf{F} \cdot \mathbf{dx}$ the quantity on the left-hand side must correspond to **relativistic energy**,

$$E := m_0 \gamma c^2. \tag{1.137}$$

It still has to be clarified which kind of energy is represented by E. Using the expression for relativistic inertial mass, Eq. (1.130), the relativistic energy can also be written in its (publicly) famous form:

$$\boxed{E = m c^2 \qquad (1.138)}$$

The physical meaning of this equation is that every energy corresponds to a mass and every mass corresponds to an energy, with the Lorentz scalar c^2 being the conversion factor.

The most startling consequence of this expression becomes evident when the special case for a particle at rest with respect to frame K is considered. Then, $||\mathbf{v}|| = 0$ and so $\gamma(||\mathbf{v}||) = 1$ and therefore $m = m_0$. In that case,

$$E = E_0 := m_0 c^2 \tag{1.139}$$

and we find that the rest mass of a body corresponds to an energy! We call E_0 the **rest energy** of a particle of rest mass m_0. This implies that energy should be convertible into rest mass (energy) and vice versa[30].

It is very important to analyze the expression Eq. (1.137) before taking any further steps. m_0 and c are Lorentz scalars, but the Lorentz factor γ is a function of velocity. We Taylor expand the Lorentz factor

[30]When formulating this equation, Einstein regarded it true from pure formal aesthetics, but considered believing in its veracity in practice as an "act of faith". It had to be confirmed by experiment, which happened in the decades to come.

about $v_0 = 0$, resulting in

$$\gamma(v) = \sum_{n=0}^{\infty} \frac{1}{n!} \left.\frac{d^n\gamma(v)}{dv^n}\right|_{v=v_0} (v - v_0)^n \tag{1.140}$$

$$= 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \mathcal{O}\left(\left(\frac{v}{c}\right)^6\right) \tag{1.141}$$

We again see that in the non-relativistic limit, $c \to \infty$, the Lorentz factor becomes 1. The above representation as a Taylor expansion is often useful in order to represent leading relativistic effects by truncating the expansion at some appropriate order.

Using the expansion Eq. (1.141) in Eq. (1.137) one obtains

$$E = m_0\, c^2 + \frac{1}{2}\, m_0 v^2 + \frac{3}{8}\, m_0 v^2\, \frac{v^2}{c^2} + \mathcal{O}\left(\left(\frac{v}{c}\right)^4\right) \tag{1.142}$$

In this form relativistic energy can be straightforwardly analyzed:

$\frac{1}{2}\, m_0 v^2$ Beginning with the known term, this represents the kinetic energy of a body with inertial mass $m = m_0$ as in non-relativistic classical mechanics.

$m_0\, c^2$ As a consequence, this term is of relativistic origin. It is evidently a Lorentz scalar, and it relates to an energy of the body independent of kinematics. It is, therefore, called the **rest energy** of the particle.

$\frac{3}{8}\, m_0 v^2\, \frac{v^2}{c^2}$ Since this contribution vanishes in the non-relativistic limit, it is also of relativistic origin and represents the leading **relativistic correction to the particle's kinetic energy**.

$\mathcal{O}\left(\left(\frac{v}{c}\right)^4\right)$ Consequently, all following terms are also relativistic corrections to the particle's kinetic energy.

1.6.1 Relativistic Energy-Momentum Relation

We will now establish an equivalent to the energy-momentum relation from non-relativistic mechanics which reads

$$T = \frac{\left(\mathbf{p}^N\right)^2}{2m_0} \quad \text{with } \mathbf{p}^N = m_0\,\mathbf{v}. \tag{1.143}$$

In the relativistic régime we realize that we can identify one relationship between relativistic momentum and relativistic energy right away. From Eqs. (1.132) and (1.138) it follows that the time-like component of the relativistic momentum four-vector is

$$p^0 \equiv mc = \frac{E}{c} \tag{1.144}$$

which means that relativistic energy appears on the time-like component of the linear momentum four-vector, such that we can recast it in another (equivalent form):

$$\{p^\mu\} = \left\{ \begin{array}{c} \frac{E}{c} \\ p^k \end{array} \right\} \quad \forall k \in \{1, \ldots, 3\} \tag{1.145}$$

The derivation of the relativistic energy-momentum relation is then just a formal exercise. We start from two equivalent forms of the momentum Lorentz scalar $p^2 = p^\mu p_\mu = p^0 p_0 + \sum_k p^k p_k$:

$$\begin{aligned} p^2 &= \frac{E^2}{c^2} - \mathbf{p}^2 \\ p^2 &= m_0^2\,\gamma^2\,c^2 - m_0^2\,\gamma^2\,\mathbf{v}^2 \end{aligned} \tag{1.146}$$

where the second relation follows from Eqs. (1.129) and (1.132). The first relation can be rewritten as

$$E^2 = c^2\left(p^2 + \mathbf{p}^2\right) \tag{1.147}$$

and inserting the second relation into it yields

$$\begin{aligned} E^2 &= c^2 \left(m_0^2 \gamma^2 c^2 - m_0^2 \gamma^2 \mathbf{v}^2 + \mathbf{p}^2 \right) \\ &= \mathbf{p}^2 c^2 + m_0^2 c^2 \left(\gamma^2 c^2 - \gamma^2 \mathbf{v}^2 \right) \\ &= \mathbf{p}^2 c^2 + m_0^2 c^2 \left(\frac{c^2}{1 - \frac{\mathbf{v}^2}{c^2}} - \frac{\mathbf{v}^2}{1 - \frac{\mathbf{v}^2}{c^2}} \right) \\ &= \mathbf{p}^2 c^2 + m_0^2 c^4 \left(\frac{c^2}{c^2 - \mathbf{v}^2} - \frac{\mathbf{v}^2}{c^2 - \mathbf{v}^2} \right) \end{aligned}$$

and so we obtain

$$E^2 = \mathbf{p}^2 c^2 + m_0^2 c^4. \tag{1.148}$$

Taking the positive square root gives

$$\boxed{E = \sqrt{\mathbf{p}^2 c^2 + m_0^2 c^4} \quad (1.149)}$$

which is known as the **relativistic energy-momentum relation**. Note that the first term under the square root is the square of linear *three*-momentum, not to be confused[31] with the scalar product of four-momentum in Eq. (1.146).

At the time of its first appearance, there was no dispute about taking into account the positive square root only, although formally the negative square root could also be permissible. After all, the notion of *negative energy* of a free particle seems queer. This point became a remarkable twist in the history of physics and will be picked up again later on in the context of relativistic quantum mechanics.

A number of interesting conclusions can be drawn from Eq. (1.149) when considering various possible cases, for which it is equally valid. The first distinction concerns the body's rest mass, m_0.

[31] Some texts use unclear notation on this point.

1.6.1.1 Particles with zero rest mass

As a first observation, we note that this case is particular to relativistic theory. The notion of a particle with zero mass makes no sense in non-relativistic mechanics. Due to the intimate relationship between energy and mass in Eq. (1.138), however, we must take this possibility seriously here.

Omitting the rest-mass term from Eq. (1.149) yields

$$E = ||\mathbf{p}||\, c. \tag{1.150}$$

However, we know that relativistic three-momentum is

$$\mathbf{p} = m_0\, \gamma(\mathbf{v})\, \mathbf{v}$$

so the energy of the "particle" seems to be zero, except if the particle is allowed to travel at the speed of light in frame K, in which case the γ factor tends to infinity! The problem obviously remains not fully resolved in classical relativistic mechanics, but a first glance at **quantum mechanics** in this context reveals an interesting connection:

Inserting de Broglie's relation ($p = \frac{h}{\lambda}$) and Planck-Einstein's relation ($E = h\nu$) which are valid for de Broglie matter waves, where h is Planck's constant, into Eq. (1.150) we obtain

$$\begin{aligned} h\nu &= \frac{h}{\lambda}\, c \\ \nu &= \frac{c}{\lambda} \end{aligned}$$

which is the well-known relationship between frequency and wavelength for waves propagating according to Maxwell's equations[32].

We conclude that the theory of **massless particles** is **necessarily a relativistic quantum theory**. We will make a first step toward this

[32]Note that trying to make the same argument based on non-relativistic momentum $p = mv$ does not lead to a consistent theory. In that case, since $\lambda = \frac{h}{mc}$, supposing propagation at the speed of light, the wavelength for a particle whose mass tends to zero becomes infinite which is in contradiction with observation. In other words, non-relativistic quantum mechanics "works" for massive particles at lower velocities.

theory in later sections (for massive particles), but a conclusive answer for massless particles will have to await the introduction of relativistic quantum field theory.

1.6.1.2 Massive Particles

For massive particles we can discuss two limiting cases:

$v << c$. From Eq. (1.142) in this approximation it follows that

$$E \approx m_0\, c^2 + \frac{\left(\mathbf{p}^N\right)^2}{2m_0} \qquad \text{for } m_0 \neq 0 \tag{1.151}$$

$v \approx c$. According to Eq. (1.132) for the linear relativistic three-momentum we can write

$$\mathbf{p}^2\, c^2 = m_0^2 \gamma^2\, \mathbf{v}^2 c^2 = m_0^2 c^4\, \frac{\mathbf{v}^2}{c^2 - \mathbf{v}^2} \tag{1.152}$$

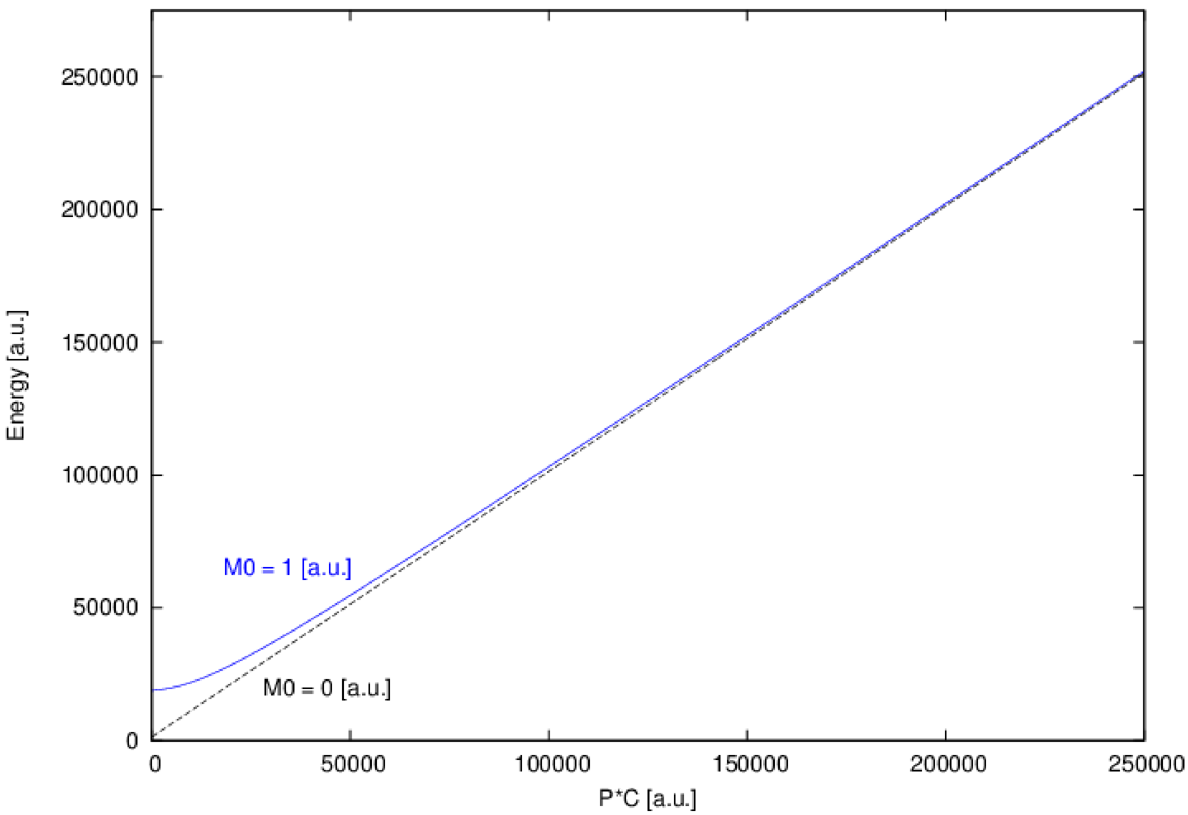
Now since $\frac{\mathbf{v}^2}{c^2 - \mathbf{v}^2} >> 1$ with the applied condition it follows that here $\mathbf{p}^2\, c^2 >> m_0^2 c^4$ and it can be approximated

$$E \approx ||\mathbf{p}||\, c \qquad \text{for } m_0 \neq 0 \tag{1.153}$$

in this limit. As a consequence, for very large relative velocities the rest energy becomes negligible as a contribution to the total relativistic energy.

Summary. Relativistic energy as a function of linear momentum for vanishing (e.g. for the photon) and non-vanishing rest mass is depicted in Fig. (1.14).

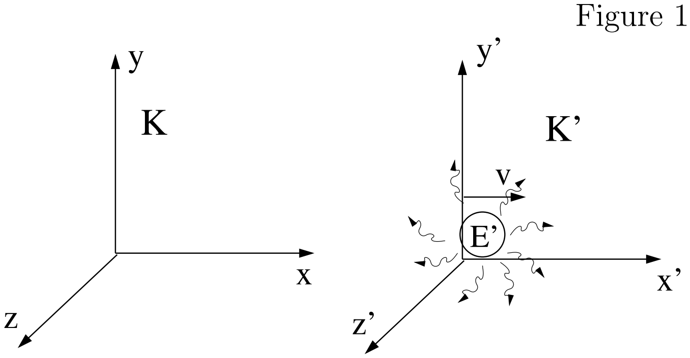
Figure 1.14:



1.6.2 Energy-Mass Equivalence: Mass Defect

In order to deepen the understanding of Eqs. (1.138) and (1.149) we will consider the following thought experiment. Be there a system at rest with respect to a frame K' that moves with velocity v relative to a laboratory frame K, Fig. (1.15).

Figure 1.15:



During a short time span $\Delta t'$ the system emits radiation of energy E' in K' symmetrically such that its total momentum in K' does not change, *i.e.*, it remains at rest in K'.

Such an emission process may, for example, occur in the formation of an atomic nucleus from its constituent nucleons (protons and neutrons). The energy is released in form of ejected particles (neutrons, α particles etc.) including kinetic energy, and radiation.

Since the energy pulse is emitted in a spherically symmetric manner its total 3-momentum in K' is zero[33]. Then the momentum four-vector for the released energy pulse takes on the following form in K and K', respectively:

Momentum four-vector in K: Momentum four-vector in K':

$$\{p^\mu\} = \left(\frac{E_{\text{rad}}}{c}, p^1 \equiv p, 0, 0\right) \qquad \{p^{\mu\prime}\} = \left(\frac{E'_{\text{rad}}}{c}, 0, 0, 0\right)$$

Of course, the momentum four-vector $\{p^{\mu\prime}\}$ has to be related with $\{p^\mu\}$ through a Lorentz transformation. In the present case we want to transform from K' to K, so we use $\mathbf{\Lambda}^{-1}(v) = \mathbf{\Lambda}(-v)$ with respect to Eq. (1.66). The transformation of momentum for the boost then reads

$$\begin{aligned} \mathbf{\Lambda}(-v) \begin{pmatrix} p^{0\prime}_{\text{rad}} \\ p^{1\prime}_{\text{rad}} \end{pmatrix} &= \begin{pmatrix} p^0_{\text{rad}} \\ p^1_{\text{rad}} \end{pmatrix} \\ \begin{pmatrix} \gamma & \frac{v}{c}\gamma \\ \frac{v}{c}\gamma & \gamma \end{pmatrix} \begin{pmatrix} \frac{E'_{\text{rad}}}{c} \\ 0 \end{pmatrix} &= \begin{pmatrix} \gamma \frac{E'_{\text{rad}}}{c} \\ \frac{v}{c}\gamma \frac{E'_{\text{rad}}}{c} \end{pmatrix} = \begin{pmatrix} p^0_{\text{rad}} \\ p^1_{\text{rad}} \end{pmatrix} \end{aligned} \tag{1.154}$$

The resulting four-vector has to be identical with the original formulation of the momentum four-vector in frame K. Let us inspect the relevant space-like component of three-momentum. We find

$$p^1_{\text{rad}} = p_{\text{rad}} = \frac{v}{c}\gamma \frac{E'_{\text{rad}}}{c} \tag{1.155}$$

which is the momentum (in K) corresponding to the energy of the radiation pulse in K'. However, the system did not change its momentum in K', due to the assumed symmetrical emission. This implies that its

[33] For instance, photons have momentum $p = \frac{h}{\lambda}$.

velocity relative to the laboratory has also not changed. What are the consequences?

At this point, it is imperative to exploit principles of symmetry. The system of our thought experiment is **isolated**. Therefore, it is **invariant to a spatial translation** and so its **total momentum in a given frame is conserved**.

The basic form of momentum in accord with Eq. (1.132) for the system is

$$p_{\mathrm{sys}} = m_{\mathrm{sys}}\,\gamma(v_{\mathrm{sys}})\,v_{\mathrm{sys}} \tag{1.156}$$

for the component of interest, where m_{sys} here is the rest mass of the system. This quantity has to be conserved due to symmetry. But Eq. (1.155) forces us to consider the momentum of the radiation pulse that is non-zero in the balance of momentum in K. And since relative velocity does not change due to the pulse, the only way to compensate is by a loss of rest mass of the system due to the emission of radiation! Formally, momentum conservation in K is thus written as

$$\begin{aligned} p_{\mathrm{sys}_{\mathrm{before}}} &= p_{\mathrm{sys}_{\mathrm{after}}} + p_{\mathrm{rad}} \\ m_{\mathrm{sys}_{\mathrm{before}}}\,\gamma\,v &= m_{\mathrm{sys}_{\mathrm{after}}}\,\gamma\,v + \frac{v}{c}\,\gamma\,\frac{E'_{\mathrm{rad}}}{c} \end{aligned} \tag{1.157}$$

from which it follows that

$$m_{\mathrm{sys}_{\mathrm{before}}} = m_{\mathrm{sys}_{\mathrm{after}}} + \frac{E'_{\mathrm{rad}}}{c^2}. \tag{1.158}$$

The energy of the radiation pulse divided by the square of the speed of light is the **rest mass lost** by the system due to the emission of the radiation pulse of energy E'. The system has, therefore, suffered a **mass defect**, defined as

$$\frac{E'_{\mathrm{rad}}}{c^2} = m_{\mathrm{sys}_{\mathrm{before}}} - m_{\mathrm{sys}_{\mathrm{after}}} \tag{1.159}$$

Note that the mass defect is Lorentz invariant because E'_{rad} refers to the rest frame of the system.

Table 1.1: Nuclear mass, mass defect and ratio for some sample nuclides

Nuclide	Mass $[u]$	Mass defect $[u]$	$\frac{\text{Mass defect } [u]}{\text{Mass } [u]}$
^{4}He	4.002603	0.029279	0.0073
^{12}C	12.000000	0.095646	0.0080
^{59}Fe	58.934874	0.540247	0.0092
^{225}Ra	225.023611	1.803782	0.0080

$$u = 931.49410242(28)\,[\frac{\text{MeV}}{c^2}]$$