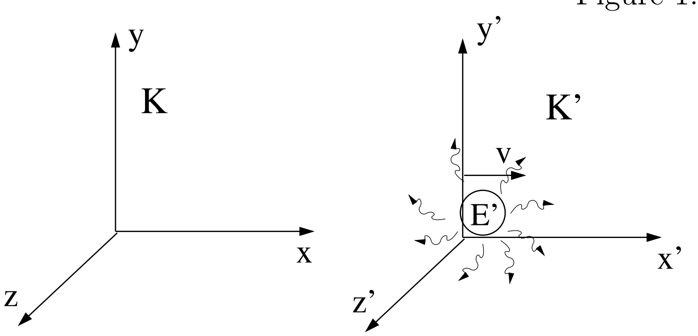
1.6.2 Energy-Mass Equivalence: Mass Defect

In order to deepen the understanding of Eqs. (1.121) and (1.132) we will consider the following thought experiment. Be there a system at rest with respect to a frame K' that moves with velocity v relative to a laboratory frame K, Fig. (1.15).

Figure 1.15:



During a short time span $\Delta t'$ the system emits radiation of energy E' in K' symmetrically such that its total momentum in K' does not change, *i.e.*, it remains at rest in K'.

Such an emission process may, for example, occur in the formation of an atomic nucleus from its constituent nucleons (protons and neutrons). The energy is released in form of ejected particles (neutrons, α particles etc.) including kinetic energy, and radiation.

Since the energy pulse is emitted in a spherically symmetric manner its total 3-momentum in K' is zero[27]. Then the momentum four-vector for the released energy pulse takes on the following form in K and K', respectively:

Momentum four-vector in K: Momentum four-vector in K':

$$\{p^\mu\} = \left(\frac{E_{\text{rad}}}{c}, p^1 \equiv p, 0, 0\right) \qquad \{p^{\mu\prime}\} = \left(\frac{E'_{\text{rad}}}{c}, 0, 0, 0\right)$$

Of course, the momentum four-vector $\{p^{\mu\prime}\}$ has to be related with $\{p^\mu\}$ through a Lorentz transformation. In the present case we want to transform from K' to K, so we use $\mathbf{\Lambda}^{-1}(v) = \mathbf{\Lambda}(-v)$ with respect to

[27]For instance, photons have momentum $p = \frac{h}{\lambda}$.

Eq. (1.52). The transformation of momentum for the boost then reads

$$\mathbf{\Lambda}(-v)\begin{pmatrix} p^{0'}_{\text{rad}} \\ p^{1'}_{\text{rad}} \end{pmatrix} = \begin{pmatrix} p^{0}_{\text{rad}} \\ p^{1}_{\text{rad}} \end{pmatrix}$$

$$\begin{pmatrix} \gamma & \frac{v}{c}\gamma \\ \frac{v}{c}\gamma & \gamma \end{pmatrix}\begin{pmatrix} \frac{E'_{\text{rad}}}{c} \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma\,\frac{E'_{\text{rad}}}{c} \\ \frac{v}{c}\,\gamma\,\frac{E'_{\text{rad}}}{c} \end{pmatrix} = \begin{pmatrix} p^{0}_{\text{rad}} \\ p^{1}_{\text{rad}} \end{pmatrix} \tag{1.137}$$

The resulting four-vector has to be identical with the original formulation of the momentum four-vector in frame K. Let us inspect the relevant space-like component of three-momentum. We find

$$p^{1}_{\text{rad}} = p_{\text{rad}} = \frac{v}{c}\,\gamma\,\frac{E'_{\text{rad}}}{c} \tag{1.138}$$

which is the momentum (in K) corresponding to the energy of the radiation pulse in K'. However, the system did not change its momentum in K', due to the assumed symmetrical emission. This implies that its velocity relative to the laboratory has also not changed. What are the consequences?

At this point, it is imperative to exploit principles of symmetry. The system of our thought experiment is **isolated**. Therefore, it is **invariant to a spatial translation** and so its **total momentum in a given frame is conserved**.

The basic form of momentum in accord with Eq. (1.117) for the system is

$$p_{\text{sys}} = m_{\text{sys}}\,\gamma(v_{\text{sys}})\,v_{\text{sys}} \tag{1.139}$$

for the component of interest, where m_{sys} here is the rest mass of the system. This quantity has to be conserved due to symmetry. But Eq. (1.138) forces us to consider the momentum of the radiation pulse that is non-zero in the balance of momentum in K. And since relative velocity does not change due to the pulse, the only way to compensate is by a loss of rest mass of the system due to the emission of radiation!

Formally, momentum conservation in K is thus written as

$$\begin{aligned} p_{\text{before}} &= p_{\text{after}} + p_{\text{rad}} \\ m_{\text{sys}_{\text{before}}} \gamma\, v &= m_{\text{sys}_{\text{after}}} \gamma\, v + \frac{v}{c} \gamma \frac{E'_{\text{rad}}}{c} \end{aligned} \tag{1.140}$$

from which it follows that

$$m_{\text{sys}_{\text{before}}} = m_{\text{sys}_{\text{after}}} + \frac{E'_{\text{rad}}}{c^2}. \tag{1.141}$$

The energy of the radiation pulse divided by the square of the speed of light is the **rest mass lost** by the system due to the emission of the radiation pulse of energy E'. The system has, therefore, suffered a **mass defect**, defined as

$$\frac{E'_{\text{rad}}}{c^2} = m_{\text{sys}_{\text{before}}} - m_{\text{sys}_{\text{after}}} \tag{1.142}$$

Note that the mass defect is Lorentz invariant because E'_{rad} refers to the rest frame of the system.

Table 1.1: Nuclear mass, mass defect and ratio for some sample nuclides

Nuclide	Mass $[u]$	Mass defect $[u]$	$\frac{\text{Mass defect } [u]}{\text{Mass } [u]}$
^{4}He	4.002603	0.029279	0.0073
^{12}C	12.000000	0.095646	0.0080
^{59}Fe	58.934874	0.540247	0.0092
^{225}Ra	225.023611	1.803782	0.0080

$u = 931.49410242(28)\,[\frac{\text{MeV}}{c^2}]$

1.7 Relativistic Kinematics of Particle Interactions

With the developments of the previous sections it is possible to study collisions of bodies in the classical (non-quantum) regime. We can even take a look at particle decays, although we here surpass the classical notion of a particle[28]. A number of general assumptions are made that lead to important simplifications:

1. **External forces have no influence on the collision process.** This implies the conservation of total energy and total momentum (before and after the process) which will be rewritten in relativistic form.

2. We here do **not consider the details of the collision process at very short range**. For example, the collision of two neutrons could be studied at the level of the constituent quarks which would require a deeper understanding of the bound state of the neutron.

Let us first set the stage by re-iterating the principles of such collisions in non-relativistic theory. Generally, primes (') denote properties after the process, no primes denote properties before the process.

1.7.1 Non-relativistic collision processes

1. $\sum_j m_j = \sum_j m'_j$
   Total mass of all intervening particles (j) is conserved. Bodies may break up in the process, but the sum of the inertial masses always remains the same.

2. $\sum_j \mathbf{p}_j = \sum_j \mathbf{p}'_j$
   All components of total momentum are conserved. This immedi-

[28] Particles generally have finite lifetimes and the decay of a particle is a complicated quantum process where the probability of decay per unit of time plays an important role.

ately follows from the fact that there are no external forces, by assumption (consider Noether's theorem).

3. Kinetic energy T may or may not be conserved.
   Of course, **total** energy is conserved, but a process may be such that kinetic energy of incident bodies is converted to some form of internal energy (such as vibrational energy or heat). The relevant distinctions are made in the following.

- **Elastic collisions.** These are characterized by a conservation of total kinetic energy
  $\sum_j T_j = \sum_j T'_j$
  There is no conversion of kinetic energy into internal energy or vice versa.

- **Inelastic collisions.** Here total kinetic energy is not conserved. We distinguish between two cases:
  1. *"Sticky"* collisions, in which kinetic energy decreases:
     $\sum_j T_j > \sum_j T'_j$
  2. *"Explosive"* collisions, in which kinetic energy increases:
     $\sum_j T_j < \sum_j T'_j$

1.7.2 Relativistic collision processes

As before, external forces are irrelevant, and the conservation laws can now be written in four-vector form. For this, we use the momentum four-vector from Eq. (1.128) before and after the collision:

$$\begin{pmatrix} \sum_j \frac{E_j}{c} \\ \sum_j \mathbf{p}_j \end{pmatrix} = \begin{pmatrix} \sum_j \frac{E'_j}{c} \\ \sum_j \mathbf{p}'_j \end{pmatrix} \tag{1.143}$$

Note that here $E_j = E_{0j} + T_j$ is the total relativistic energy, the sum of rest energy and relativistic kinetic energy of the particle.

Also as before, kinetic energy may or may not be conserved, depending on whether a conversion into or from internal energy is characteristic of the collision. The corresponding distinctions are in this context made as follows:

- **Elastic collisions.** These are characterized by a conservation of total rest and kinetic energy
  $\sum_j E_{0j} = \sum_j E_{0j}'$
  $\sum_j T_j = \sum_j T_j'$
  There is no conversion of kinetic energy into internal (rest) energy or vice versa.

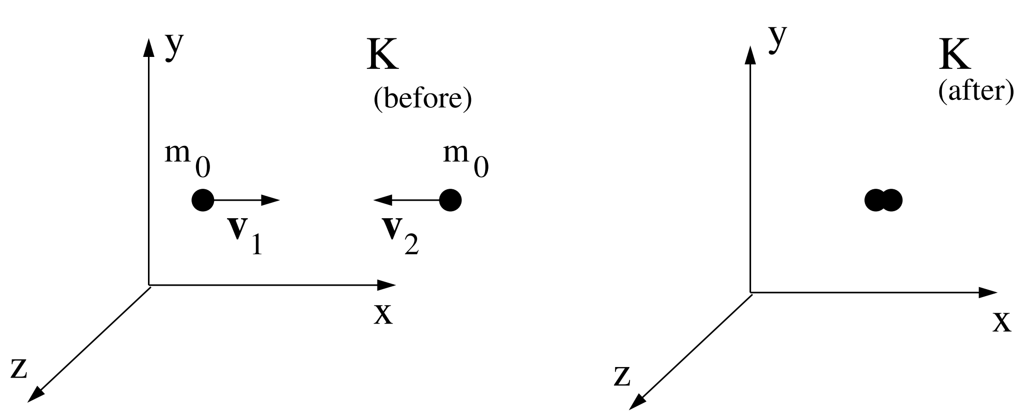
- **Inelastic collisions.** Here neither total kinetic energy nor internal (rest) energy are conserved:
  1. *"Sticky"* collisions, in which the decrese of total kinetic energy is accompanied by an increase of total rest energy:
     $\sum_j T_j > \sum_j T_j'$
     $\sum_j E_{0j} < \sum_j E_{0j}'$
     This is the typical scenario in collider physics where very heavy particles are created from lighter particles using their great incident kinetic energies.
  2. *"Explosive"* collisions, where it is the other way around:
     $\sum_j T_j < \sum_j T_j'$
     $\sum_j E_{0j} > \sum_j E_{0j}'$

The discussion of several exemplifying cases is useful for two reasons: First, examples make the peculiarities of relativistic kinematics very clear. Second, the treatment of concrete cases reveals the sometimes particular techniques of calculation.

1.7.2.1 "Sticky" Two-body Frontal Collision

The first example is very simple. We imagine two particles (or pieces of clay) of equal rest mass m_0 at high velocity, $v = \frac{3}{5}\,c$, in frontal collision under the sole assumption that the two bodies form a single body after collision (extreme sticky collision).

Figure 1.16:



We wish to determine what the rest mass, M, of the resulting body will be.

We start out be giving the 3-momentum for the two particles in K (the lower index is here a particle index)

$$\begin{aligned}\mathbf{p}_1 &= m_0\,\gamma_1\,\mathbf{v}_1\\ \mathbf{p}_2 &= m_0\,\gamma_2\,\mathbf{v}_2\end{aligned}$$

Since by assumption $\mathbf{v}_2 = -\mathbf{v}_1$ it follows that $\gamma_1 = \gamma_2$ and, therefore,

$$\mathbf{p}_2 = -\mathbf{p}_1 \tag{1.144}$$

Momentum conservation now dictates that

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_M$$

From this and Eq. (1.144) it follows that

$$\mathbf{p}_M = \mathbf{0} \Rightarrow \gamma_M = 1 \Rightarrow M = M_0$$

If the relativistic mass of the resulting body equals its rest mass in frame K, then its relativistic energy is

$$E_M = M_0\, c^2$$

We now invoke the conservation of energy, i.e., the relativistic energies before and after the collision have to be identical:

$$E_1 + E_2 = M_0\, c^2 \tag{1.145}$$

Since $\gamma_1 = \gamma_2 := \gamma$ we have

$$E_1 = m_0\, \gamma\, c^2 = E_2$$

The γ factor for the incident particles is calculated as $\gamma = \frac{1}{\sqrt{1-\frac{9}{25}\frac{c^2}{c^2}}} = \frac{1}{\sqrt{\frac{16}{25}}} = \frac{5}{4}$ and so the total energy becomes

$$E_1 + E_2 = 2m_0\, \gamma\, c^2 = \frac{5}{2}\, m_0\, c^2$$

Therefore, with Eq. (1.145),

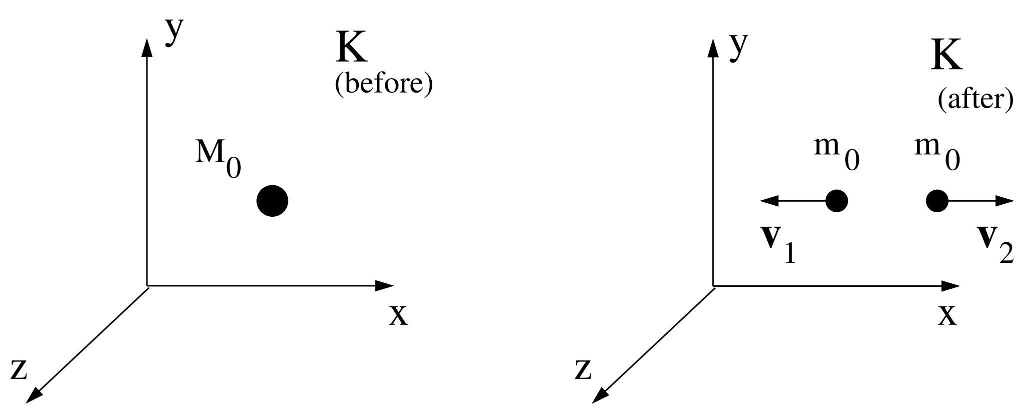
$$\begin{aligned} M_0\, c^2 &= \frac{5}{2}\, m_0\, c^2 \\ \Leftrightarrow M_0 &= \frac{5}{2}\, m_0 \end{aligned} \tag{1.146}$$

The essential observation is that the final rest mass is $\frac{5}{2}\, m_0$ which **surpasses** the initial rest mass which is $\frac{4}{2}\, m_0$. In other words, since the kinetic energy of the created particle is zero, the **complete kinetic energy of the incident particles has been converted into rest energy** of the resulting particle!

1.7.3 Spontaneous Two-body Decay

Suppose now that the initial particle has a finite lifetime. Under the assumption that it decays into two particles of equal rest mass, what will the velocities of these be in K?

Figure 1.17:



Similar to the above example, we start out from momentum conservation:

$$\begin{aligned} \mathbf{p}_M &= \mathbf{0} = \mathbf{p}_1 + \mathbf{p}_2 \\ \Rightarrow \mathbf{p}_1 &= -\mathbf{p}_2 \end{aligned} \tag{1.147}$$

and the created particles necessarily are ejected "back to back". Since their rest masses are assumed to be equal,

$$\begin{aligned} \gamma_1\,\mathbf{v}_1 &= -\gamma_2\,\mathbf{v}_2 \\ \Rightarrow \frac{1}{\gamma_2}\,||\mathbf{v}_1|| &= \frac{1}{\gamma_1}\,||\mathbf{v}_2|| \\ \Rightarrow \left(c^2 - v_2^2\right) v_1^2 &= \left(c^2 - v_1^2\right) v_2^2 \\ \Leftrightarrow v_1^2 &= v_2^2 \\ \Rightarrow \gamma_1 &= \gamma_2 \end{aligned} \tag{1.148}$$

The velocities and gamma factors of the two resulting particles are identical, not surprisingly. Due to energy conservation

$$\begin{aligned}
M_0 c^2 &= 2m_0\,\gamma\,c^2 \\
\Leftrightarrow \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} &= \frac{M_0}{2m_0} \\
\Rightarrow \frac{c^2}{c^2-v^2} &= \frac{M_0^2}{(2m_0)^2} \\
\Leftrightarrow c^2 - v^2 &= \frac{(2m_0)^2\,c^2}{M_0^2} \\
\Leftrightarrow v &= c\sqrt{1-\left(\frac{2m_0}{M_0}\right)^2} \qquad (1.149)
\end{aligned}$$

where only the positive root is taken since we understand $v = ||\mathbf{v}||$. Thus, we find the condition

$$M_0 \geq 2m_0 \qquad (1.150)$$

since otherwise velocity becomes imaginary. This result implies that there is a **threshold energy** E_T, a minimum energy of the incident particle, defined as

$$E_T = M_0\,c^2 := 2m_0 c^2 \qquad (1.151)$$

for the two-body decay into two particles of identical rest mass to take place. A surplus of energy in the form of initial kinetic energy is not an obstacle.

This result is of general importance as it helps exclude possible decay processes in particle physics based on the rest masses of the involved particles alone. A good example is the deuteron d with $m_d = 1875.6\,[\frac{\text{MeV}}{c^2}]$. The presumed decay process $d \longrightarrow p+n$ is kinematically impossible since $m_p + m_n = 1877.9\,[\frac{\text{MeV}}{c^2}]$ which makes the deuteron a stable particle.

1.7.4 Pion Decay and Special Methods

As a final example which also introduces a clever technique for solving problems of this kind, we consider the decay of the π-meson (or simply pion) π^- ($\equiv d\overline{u}, C = -1, m_{\pi^-} = 139.57\,[\frac{\text{MeV}}{c^2}]$) into a muon μ^- ($C = -1, m_{\mu^-} = 105.66\,[\frac{\text{MeV}}{c^2}]$) and a muon-antineutrino[29] $\overline{\nu}_\mu$ ($C = 0, m_{\overline{\nu}_\mu} \approx 0$), i.e.,

$$\pi^- \longrightarrow \mu^- + \overline{\nu}_\mu \tag{1.152}$$

Neutrinos and their antiparticles have — for all practical purposes — zero rest mass and thus propagate with the speed of light. The particular question of the **velocity of the resulting muon** is interesting here. Moreover, we want to express this velocity solely in terms of the known parameters of the problem which are the rest masses and the speed of light.

First, a useful little theorem can be derived. Three-momentum and energy of a given particle in the lab frame are

$$\begin{aligned} \mathbf{p} &= m_0\,\gamma\,\mathbf{v} \\ E &= m_0\,\gamma\,c^2 \end{aligned}$$

and so by division of these two identities

$$\begin{aligned} \frac{\mathbf{p}}{E} &= \frac{\mathbf{v}}{c^2} \\ \Leftrightarrow \mathbf{v} &= \frac{\mathbf{p}c^2}{E} \end{aligned} \tag{1.153}$$

In other words, if the momentum and energy of a particle are known, so is its velocity. We write this expression specifically as a norm for the muon quantities

$$||\mathbf{v}_\mu|| = \frac{||\mathbf{p}_\mu||c^2}{E_\mu} \tag{1.154}$$

and evaluate it.

[29] It has to be an *anti*-neutrino due to conservation laws that will be discussed further down the road.

An elegant trick to obtain the required momentum and energy is to work from the Minkowski-space scalar product four-conservation. Four-conservation is for the present case written in terms of the relevant momentum four-vectors:

$$\begin{aligned} p_\pi = \begin{pmatrix} p_\pi^0 = \frac{E_\pi}{c} \\ \mathbf{p}_\pi \end{pmatrix} &= p_\mu + p_{\overline{\nu}} = \begin{pmatrix} p_\mu^0 + p_{\overline{\nu}}^0 = \frac{E_\mu}{c} + \frac{E_{\overline{\nu}}}{c} \\ \mathbf{p}_\mu + \mathbf{p}_{\overline{\nu}} \end{pmatrix} \\ \Leftrightarrow p_{\overline{\nu}} &= p_\pi - p_\mu \end{aligned} \tag{1.155}$$

Based on the last identity we calculate the scalar product of the anti-neutrino four-momentum with itself:

$$p_{\overline{\nu}}^2 = p_\pi^2 + p_\mu^2 - 2p_\pi \cdot p_\mu \tag{1.156}$$

The various terms in Eq. (1.156) are now calculated one by one.

1. Since the problem is treated in the rest frame of the pion its three-momentum is zero and so, using the relativistic energy-momentum relation Eq. (1.132), $E_\pi = \sqrt{\mathbf{p}_\pi^2 c^2 + m_\pi^2 c^4} = m_\pi c^2 = E_{0_\pi}$. Now we do the four-scalar product

$$p_\pi^2 = \frac{E_{0_\pi}^2}{c^2} = \frac{m_\pi^2 c^4}{c^2} = m_\pi^2 c^2 \tag{1.157}$$

2. The created muon cannot be expected to have zero momentum in K. The corresponding calculation for the next term is, therefore,

$$p_\mu^2 = \frac{E_\mu^2}{c^2} - ||\mathbf{p}_\mu||^2 = \frac{||\mathbf{p}_\mu||^2 c^2 + m_\mu^2 c^4}{c^2} - ||\mathbf{p}_\mu||^2 = m_\mu^2 c^2 \tag{1.158}$$

3. With the above findings the Minkowski scalar product between muon and pion four-momenta is easily obtained as

$$p_\pi \cdot p_\mu = \frac{E_\pi}{c}\frac{E_\mu}{c} - \mathbf{p}_\pi \cdot \mathbf{p}_\mu = \frac{m_\pi c^2}{c}\frac{E_\mu}{c} - 0 = m_\pi E_\mu \tag{1.159}$$

   By theory (see Eq. (1.63)) $p_\pi \cdot p_\mu = (p_\pi)^\lambda (p_\mu)_\lambda$ is a Lorentz scalar, as will become evident in the following.

4. Finally, the left-hand side of Eq. (1.156) gives, using the energy-momentum relation for massless particles Eq. (1.133),

$$p_{\overline{\nu}}^2 = \frac{E_{\overline{\nu}}^2}{c^2} - ||\mathbf{p}_{\overline{\nu}}||^2 = \frac{||\mathbf{p}_{\overline{\nu}}||^2\, c^2}{c^2} - ||\mathbf{p}_{\overline{\nu}}||^2 = 0 \qquad (1.160)$$

With the results from calculations 1 . . . 4 Eq. (1.156) becomes

$$\begin{aligned} 0 &= m_\pi^2 c^2 + m_\mu^2 c^2 - 2m_\pi\, E_\mu \\ \Leftrightarrow E_\mu &= \frac{m_\pi^2 + m_\mu^2}{2m_\pi}\, c^2 \end{aligned} \qquad (1.161)$$

and so we have obtained an expression for the first ingredient required to evaluate Eq. (1.154). A similar calculation leads to the momentum of the muon:

$$\begin{aligned} p_\mu &= p_\pi - p_{\overline{\nu}} \\ \Rightarrow p_\mu^2 &= p_\pi^2 + p_{\overline{\nu}}^2 - 2p_\pi \cdot p_{\overline{\nu}} \\ \Leftrightarrow m_\mu^2 c^2 &= m_\pi^2 c^2 + 0 - 2\,\frac{E_\pi}{c}\,\frac{E_{\overline{\nu}}}{c} - \mathbf{p}_\pi \cdot \mathbf{p}_{\overline{\nu}} \end{aligned} \qquad (1.162)$$

$\mathbf{p}_\pi = \mathbf{0}$ and thus $E_\pi = m_\pi c^2$. Furthermore, since the anti-neutrino is massless, $E_{\overline{\nu}} = ||\mathbf{p}_{\overline{\nu}}||\, c = ||\mathbf{p}_\mu||\, c$. The latter identity follows from momentum conservation

$$\begin{aligned} \mathbf{p}_\pi &= \mathbf{0} = \mathbf{p}_\mu + \mathbf{p}_{\overline{\nu}} \\ \Leftrightarrow \mathbf{p}_\mu &= -\mathbf{p}_{\overline{\nu}} \\ \Rightarrow ||\mathbf{p}_\mu|| &= ||\mathbf{p}_{\overline{\nu}}|| \end{aligned} \qquad (1.163)$$

Putting all of this together yields for Eq. (1.162)

$$\begin{aligned} m_\mu^2 c^2 &= m_\pi^2 c^2 - 2\, m_\pi\, ||\mathbf{p}_\mu|| c \\ \Leftrightarrow ||\mathbf{p}_\mu|| &= \frac{-m_\mu^2 + m_\pi^2}{2\, m_\pi}\, c \end{aligned} \qquad (1.164)$$

The velocity of the created muon results from the combination of Eqs. (1.154), (1.161), and (1.164) to be

$$||\mathbf{v}_\mu|| = \frac{\frac{-m_\mu^2+m_\pi^2}{2\,m_\pi}\,c^3}{\frac{m_\pi^2+m_\mu^2}{2m_\pi}\,c^2} = \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2}\,c \tag{1.165}$$

which depends only on the rest masses of the involved particles.

Using these known masses, $v_\mu \approx 0.271\,c$.