1.3.3 Relativistic Version of Newton's Equation of Motion

We can now proceed to formulating the fundamental law of dynamics in the domain of special relativity. For this, a further definition is required.

We do not know at this point, how mass behaves under Lorentz transformation, but this will be derived. So we **define** that m_0 be the **mass of a particle in its rest frame**, which is then by definition a Lorentz scalar. This means that rest mass of a particle never changes, just like proper time is invariant of the state of movement of a relative observer.

In view of Newton's equation of motion from non-relativistic theory,

$$m\,\mathbf{a} = \sum_i \mathbf{F}_i = \mathbf{F} \tag{1.102}$$

where there is only one "type" of mass of a particle which never changes, irrespective of any state of motion and force is a three-vector, we **define**

$$\boxed{m_0\, b^\mu = K^\mu \qquad (1.103)}$$

to be the μth SpaceTime component of the **relativistic fundamental law of dynamics**. $m_0 b^\mu$ necessarily transforms like a contravariant four-vector since m_0 is a Lorentz scalar. Likewise, K^μ is a component of a contravariant four-vector which verfies the homogeneity of the equation in that respect. $\{K^\mu\}$ is called **Minkowski's force four-vector**.

Obviously, the guiding principles for this definition are the replacement of three- by four-vectors and the conservation of the form of the equation of motion in the relativistic domain. Nevertheless, the consequences of this formulation have to conform with experimental tests.

For the analysis of the new equation we resort to relations developed earlier. In subsection 1.3.1 it has been shown that for the proper time differential $d\tau = \frac{1}{\gamma(v)}\,dt$. Furthermore, using Eq. (1.99), Eq. (1.103) yields

$$m_0 \frac{du^\mu}{d\tau} = m_0\gamma(v)\,\frac{du^\mu}{dt} = K^\mu \tag{1.104}$$

1.3.3.1 Space-Like Part of the Equation of Motion

Using Eq. (1.97) the **space-like part** of Eq. (1.104) becomes

$$\begin{aligned} m_0\,\gamma(v)\,\frac{d}{dt}\left(\gamma(v)\,v_k\right) &= K^k \\ \frac{d}{dt}\left[\gamma(v)\,m_0\,v_k\right] &= \frac{1}{\gamma(v)}\,K^k \end{aligned} \tag{1.105}$$

This equation is reminiscent of the temporal change of the quantity of movement, linear momentum, in non-relativistic mechanics which is related to the force acting on the particle:

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} \tag{1.106}$$

The immediate implication is that we can postulate how relativistic linear momentum should be formulated. However, before doing so, let us complete the discussion of Minkowski's force four-vector.

First, in accord with this finding, it is postulated that

$$K^k :\equiv \gamma(v)\,F_k \tag{1.107}$$

for the relation between four- and three-force components. Again, components K^k and F_k become equivalent in the non-relativistic limit of relativistic theory (also notation-wise, considering our convention).

1.3.3.2 Time-Like Part of the Equation of Motion

Now it is time to derive the time-like component of the Minkowski force vector. For this, we multiply the relativistic fundamental law of dynamics, Eq. (1.103), by $g_{\mu\nu}\, u^\nu$ (including summation according to Einstein convention) and obtain

$$\begin{aligned} m_0\, b^\mu &= K^\mu \\ m_0\, g_{\mu\nu}\, u^\nu\, b^\mu &= g_{\mu\nu}\, u^\nu\, K^\mu \\ m_0\, u^\nu\, g_{\nu\mu}\, b^\mu &= K^\mu\, g_{\mu\nu}\, u^\nu \\ m_0\, u_\mu\, b^\mu &= K^0\, u^0 - \sum_{k=1}^{3} K^k\, u^k \end{aligned}$$

We can now readily use earlier results. For one thing, $u_\mu\, b^\mu = 0$ according to Eq. (1.101). For the other, the definition in Eq. (1.107) and the form of the velocity four-vector in Eq. (1.97) entail the identity $\sum_{k=1}^{3} K^k\, u^k = \gamma(v)\mathbf{F}\cdot\gamma(v)\mathbf{v}$, and so we arrive at

$$K^0\, u^0 - \gamma^2\, \mathbf{F}\cdot\mathbf{v} = 0 \tag{1.108}$$

$$K^0\, \gamma\, c = \gamma^2\, \mathbf{F}\cdot\mathbf{v}$$

$$K^0 = \frac{\gamma}{c}\, \mathbf{F}\cdot\mathbf{v} \tag{1.109}$$

In summary, Minkowski's force four-vector is thus written as

$$\{K^\mu\} = \left\{ \begin{array}{c} \frac{\gamma}{c}\,\mathbf{F}\cdot\mathbf{v} \\ \gamma F_k \end{array} \right\} \quad \forall k \in \{1,\ldots,3\} \tag{1.110}$$

In the non-relativistic limit we find:

$$\begin{aligned} \lim_{c\to\infty} K^0 &= 0 \\ \lim_{c\to\infty} K^k &= F_k. \end{aligned}$$

With the above equations (1.100) and (1.110) Newton's equation of motion can be rewritten in relativistic form. The full relativistic mechanical equation of motion is thus given as

$$m_0 \, b^\mu = K^\mu \quad (1.111)$$

$$m_0 \, \frac{\gamma^4}{c} \left(\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} \right) = \frac{\gamma}{c} \, \mathbf{F} \cdot \mathbf{v}$$

$$m_0 \left[\frac{\gamma^4}{c^2} \, v_k \left(\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} \right) + \gamma^2 a_k \right] = \gamma F_k$$

where the first line is the covariant notation with contravariant components of four-vectors, the second line is the time-like component and the third line are the space-like components ($\forall k$) of the equation of motion.

It is quite evident that this equation describes physics different from Newton's law. Focussing on the space-like part and neglecting the first term on the l.h.s. leads to

$$m_0 \gamma a_k = F_k.$$

This approximate equation differs from the non-relativistic law of motion through the presence of the γ factor. Moreover, the first contribution scales differently with γ and cannot compensate for this change as simple acceleration problems (like a particle in an external electric field) demonstrate.

1.4 Relativistic Formulation of Classical Electrodynamics

Let us briefly look at the reformulation of classical electrodynamics in terms of the new quantities for relativistic theory, four-tensors and Lorentz scalars.

1.4.1 A Digression on Units

A few comments concerning the choice of units for the following sections are in place. We replace the standard S.I. units in the following by Gaussian-based S.I. units which corresponds to making the replacements for the electric and magnetic field as well as the charge density

$$\begin{aligned} \mathbf{E} &\rightarrow \frac{\mathbf{E}}{\sqrt{4\pi\varepsilon_0}} \\ \mathbf{B} &\rightarrow \mathbf{B}\sqrt{\frac{\mu_0}{4\pi}} \\ \rho &\rightarrow \rho\sqrt{4\pi\varepsilon_0} \end{aligned} \tag{1.112}$$

Then, for instance, Faraday's law of induction

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad (S.I.) \tag{1.113}$$

becomes

$$\begin{aligned} \frac{1}{\sqrt{4\pi\varepsilon_0}}\boldsymbol{\nabla} \times \mathbf{E} &= -\sqrt{\frac{\mu_0}{4\pi}}\frac{\partial \mathbf{B}}{\partial t} \\ \boldsymbol{\nabla} \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad (\text{Gaussian } S.I.) \end{aligned} \tag{1.114}$$

with $\sqrt{\mu_0\varepsilon_0} = \frac{1}{c}$. We furthermore see that in the Gaussian system the electric and magnetic fields have the **same** units.

Since $q \propto \rho$, Coulomb's law becomes

$$\mathbf{F}_{12} = \frac{q_1 q_2}{r_{12}^2}\mathbf{e}_{12} \tag{1.115}$$

We will also be interested in Ampère's law:

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{1.116}$$

With the replacements defined in Eq. (1.112) Ampère's law is written as

$$\begin{aligned}
\sqrt{\frac{\mu_0}{4\pi}}\,\boldsymbol{\nabla} \times \mathbf{B} &= \mu_0 \sqrt{4\pi\varepsilon_0}\,\mathbf{J} + \frac{\mu_0\varepsilon_0}{\sqrt{4\pi\varepsilon_0}}\,\frac{\partial \mathbf{E}}{\partial t} \\
\sqrt{\mu_0\varepsilon_0}\,\boldsymbol{\nabla} \times \mathbf{B} &= 4\pi\mu_0\varepsilon_0\,\mathbf{J} + \mu_0\varepsilon_0\,\frac{\partial \mathbf{E}}{\partial t} \\
\boldsymbol{\nabla} \times \mathbf{B} &= \frac{1}{c}\left(4\pi\mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}\right) \qquad (1.117)
\end{aligned}$$

1.4.2 Continuity Equation

This is a simple – but important – example to begin with. The continuity equation of electrodynamics is derived from applying the theorem of Gauss and Ostrodradsky[26] to the relation between charged current density flux through a surface and change of total charge in a volume delimited by that surface:

$$\frac{\partial \rho(\mathbf{r}, t)}{\partial t} = -\boldsymbol{\nabla} \cdot \mathbf{J}(\mathbf{r}, t) \tag{1.118}$$

Note that the continuity equation takes the same form in the Gaussian unit system.

Before proceeding, we introduce a new quantity, the four-vector of charged current density

$$\{J^\mu\} = \left\{ \begin{array}{c} J^0 = c\rho \\ J^k \equiv J_k \end{array} \right\}^{22} \quad \forall k \in \{1, \ldots, 3\}. \tag{1.119}$$

[26] Which can be proven for **any** vector field on purely geometric grounds.

The above form of the four-vector seems to be an *ad hoc* assumption at this stage. However, considering that classical current density is charge density times velocity, we see that Eq. (1.119) has the same structure as the velocity four vector in Eq. (1.97). Note that the physical dimension of the time-like component of this four-vector is $\left[\frac{Q}{L^3} \times \frac{L}{T}\right]$, *i.e.*, charge density times velocity, which is the same as $\dim[\vec{J}]$.

We now reformulate Eq. (1.118), using $\partial_0 = \frac{\partial}{\partial x^0} = \frac{1}{c}\frac{\partial}{\partial t}$:

$$\begin{aligned}
\frac{1}{c}\frac{\partial}{\partial t}c\rho + \nabla \cdot \mathbf{J} &= 0 \\
\frac{\partial}{\partial x^0}c\rho + \sum_{j=1}^{3}\frac{\partial}{\partial x^j}J^j &= 0 \\
\partial_0 J^0 + \sum_{j=1}^{3}\partial_j J^j &= 0 \\
\partial_\mu J^\mu &= 0 \qquad (1.120)
\end{aligned}$$

Eq. (1.120) is manifestly written in covariant form, with either side of the equation a Lorentz scalar. In other words, the continuity equation of electrodynamics is Lorentz invariant, even though the individual terms (∂_μ and J^μ) transform as four-vectors. This also means that charge conservation is independent of the inertial frame in which it is regarded.

It is an instructive exercise to show that the continuity equation of electrodynamics is not Galilei invariant. This can be achieved, for example, by using the Galilei boost transformation in Eqs. (12) and the elucidations from section 0.2.

[22] Note that the same symbol "J" is used for denoting four-vectors and three-vectors, so the identity $J^k \equiv J_k$ means the equivalence between the contravariant four-vector component J^k and J_k in non-relativistic notation, NOT the equivalence with the covariant components of J.

1.4.3 Maxwell's Equations

We begin by defining a four-vector of the electromagnetic potential:

$$\{A^\mu\} = \left\{ \begin{array}{c} A^0 \\ A^k \end{array} \right\} \equiv \left\{ \begin{array}{c} V \\ A_k \end{array} \right\} \quad \forall k \in \{1, \ldots, 3\} \tag{1.121}$$

where V is the usual scalar potential and $\mathbf{A}$ is the vector potential (where again the same symbol is used in covariant and non-relativistic notation).

Further, we require what is called the **tensor of the electromagnetic field**

$$F^{\mu\nu} := \partial^\mu A^\nu - \partial^\nu A^\mu \tag{1.122}$$

F is here defined as a rank-2 four-tensor with two contravariant indices. Covariant (or mixed) components of this tensor can be derived by using the metric tensor. Evidently, the left- and the right-hand side of Eq. (1.122) transform in the same manner under Lorentz transformation, but of course F is not a Lorentz scalar.

The elements of the field tensor shall be determined for two examples.

$$\begin{aligned} F^{01} &= \partial^0 A^1 - \partial^1 A^0 = \partial_0 A^1 + \partial_1 A^0 \\ &= \frac{\partial}{\partial x^0} A^1 + \frac{\partial}{\partial x^1} A^0 = \frac{\partial}{\partial x} V + \frac{1}{c}\frac{\partial}{\partial t} A_x \\ &= -\left(-\boldsymbol{\nabla} V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right)_x \\ &= -E_x \end{aligned}$$

using a combination of the structure equations of electrodynamics in the last step[27]. As a second example consider

$$\begin{aligned} F^{12} &= \partial^1 A^2 - \partial^2 A^1 = \frac{\partial}{\partial x_1} A^2 - \frac{\partial}{\partial x_2} A^1 \\ &= -\frac{\partial}{\partial x^1} A^2 + \frac{\partial}{\partial x^2} A^1 = -\frac{\partial}{\partial x} A_y + \frac{\partial}{\partial y} A_x \\ &= -\left(\boldsymbol{\nabla} \times \mathbf{A}\right)_z \\ &= -B_z \end{aligned}$$

and so on for the remaining elements. Obviously, $F^{\mu\mu} = 0$, and so the field tensor takes on the form

$$\{F^{\mu\nu}\} \equiv \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} . \tag{1.124}$$

We now postulate that the **inhomogeneous Maxwell equations (Gauss's and Ampère's law)** can be written in a very elegant and compact form:

$$\boxed{\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu \qquad (1.125)}$$

This shall be verified by two examples. First, setting $\nu = 0$:

$$\begin{aligned} \partial_0 F^{00} + \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} &= \frac{4\pi}{c} J^0 \\ 0 + \frac{\partial}{\partial x} E_x + \frac{\partial}{\partial y} E_y + \frac{\partial}{\partial z} E_z &= \frac{4\pi}{c} c\rho \\ \boldsymbol{\nabla} \cdot \mathbf{E} &= 4\pi\rho \end{aligned} \tag{1.126}$$

[27] Writing the electric field as

$$\begin{aligned} \mathbf{E} &= -\boldsymbol{\nabla} V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \\ \boldsymbol{\nabla} \times \mathbf{E} &= -\boldsymbol{\nabla} \times \boldsymbol{\nabla} V - \frac{1}{c}\frac{\partial}{\partial t} \boldsymbol{\nabla} \times \mathbf{A} \\ \boldsymbol{\nabla} \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \end{aligned} \tag{1.123}$$

leads to the Maxwell-Faraday equation.

which is nothing else than Gauss's law in differential form[28]. Now we set $\nu = 1$:

$$\begin{aligned} \partial_0 F^{01} + \partial_1 F^{11} + \partial_2 F^{21} + \partial_3 F^{31} &= \frac{4\pi}{c} J^1 \\ \frac{1}{c}\frac{\partial}{\partial t}(-E_x) + 0 + \frac{\partial}{\partial y} B_z + \frac{\partial}{\partial z}(-B_y) &= \frac{4\pi}{c} J_x \end{aligned} \tag{1.127}$$

$$\left(\boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \right)_x = \frac{4\pi}{c} J_x \tag{1.128}$$

which is one cartesian component of Ampère's law.

The power in the formulation of Eq. (1.125) lies in the fact that it is written in a homogeneous way in terms of four-vectors ($\partial_\mu F^{\mu\nu}$ transforms like a contravariant four-vector) which makes it Lorentz covariant, *i.e.*, form invariant with respect to Lorentz transformations. This is by no means obvious in the original formulation of Maxwell's equations.

[28]It is easily checked that the S.I. form of Gauss's law $\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$ becomes Eq. (1.126) upon making the replacements from Eqs. (1.112).