1.4 Relativistic Formulation of Classical Electrodynamics

Let us briefly look at the reformulation of classical electrodynamics in terms of the new quantities for relativistic theory, four-tensors and Lorentz scalars.

1.4.1 A Digression on Units

A few comments concerning the choice of units for the following sections are in place. We replace the standard S.I. units in the following by Gaussian-based S.I. units which corresponds to making the replacements for the electric and magnetic field as well as the charge density

$$\begin{aligned} \mathbf{E} &\rightarrow \frac{\mathbf{E}}{\sqrt{4\pi\varepsilon_0}} \\ \mathbf{B} &\rightarrow \mathbf{B}\sqrt{\frac{\mu_0}{4\pi}} \\ \rho &\rightarrow \rho\sqrt{4\pi\varepsilon_0} \end{aligned} \tag{1.91}$$

Then, for instance, Faraday's law of induction

$$\boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \qquad (S.I.) \tag{1.92}$$

becomes

$$\begin{aligned} \frac{1}{\sqrt{4\pi\varepsilon_0}}\boldsymbol{\nabla}\times\mathbf{E} &= -\sqrt{\frac{\mu_0}{4\pi}}\frac{\partial\mathbf{B}}{\partial t} \\ \boldsymbol{\nabla}\times\mathbf{E} &= -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \qquad (\text{Gaussian } S.I.) \end{aligned} \tag{1.93}$$

with $\sqrt{\mu_0\varepsilon_0} = \frac{1}{c}$. We furthermore see that in the Gaussian system the electric and magnetic fields have the **same** units.

Since $q \propto \rho$, Coulomb's law becomes

$$\mathbf{F}_{12} = \frac{q_1 q_2}{r_{12}^2}\mathbf{e}_{12} \tag{1.94}$$

We will also be interested in Ampère's law:

$$\boldsymbol{\nabla}\times\mathbf{B}=\mu_0\mathbf{J}+\mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t} \tag{1.95}$$

With the replacements defined in Eq. (1.91) Ampère's law is written as

$$\begin{aligned}
\sqrt{\frac{\mu_0}{4\pi}}\,\boldsymbol{\nabla}\times\mathbf{B} &= \mu_0\sqrt{4\pi\varepsilon_0}\,\mathbf{J}+\frac{\mu_0\varepsilon_0}{\sqrt{4\pi\varepsilon_0}}\frac{\partial\mathbf{E}}{\partial t}\\
\sqrt{\mu_0\varepsilon_0}\,\boldsymbol{\nabla}\times\mathbf{B} &= 4\pi\mu_0\varepsilon_0\,\mathbf{J}+\mu_0\varepsilon_0\,\frac{\partial\mathbf{E}}{\partial t}\\
\boldsymbol{\nabla}\times\mathbf{B} &= \frac{1}{c}\left(4\pi\mathbf{J}+\frac{\partial\mathbf{E}}{\partial t}\right) \qquad (1.96)
\end{aligned}$$

1.4.2 Continuity Equation

This is a simple – but important – example to begin with. The continuity equation of electrodynamics is derived from applying the theorem of Gauss and Ostrodradsky[16] to the relation between charged current density flux through a surface and change of total charge in a volume delimited by that surface:

$$\frac{\partial\rho(\mathbf{r},t)}{\partial t}=-\boldsymbol{\nabla}\cdot\mathbf{J}(\mathbf{r},t) \tag{1.97}$$

Note that the continuity equation takes the same form in the Gaussian unit system.

Before proceeding, we introduce a new quantity, the four-vector of charged current density

$$\boxed{\{J^\mu\}=\left\{\begin{array}{c}J^0=c\rho\\ J_k\end{array}\right\}\quad\forall k\in\{1,\dots,3\}.\ (1.98)}$$

[16]Which can be proven for **any** vector field on purely geometric grounds

The above form of the four-vector seems to be an *ad hoc* assumption at this stage. However, as will become clear in the following, it follows from the Lorentz invariance of the conservation of charge. Note that the physical dimension of the time-like component of this four-vector is $\left[\frac{Q}{L^3} \times \frac{L}{T}\right]$, *i.e.*, charge density times velocity, which is the same as dim$[J]$.

We now reformulate Eq. (1.97):

$$\begin{aligned}
\frac{1}{c}\frac{\partial}{\partial t}c\rho + \boldsymbol{\nabla}\cdot\mathbf{J} &= 0 \\
\frac{\partial}{\partial x^0}c\rho + \sum_{j=1}^{3}\frac{\partial}{\partial x^j}J^j &= 0 \\
\partial_0 J^0 + \sum_{j=1}^{3}\partial_j J^j &= 0 \\
\partial_\mu J^\mu &= 0 \qquad (1.99)
\end{aligned}$$

Eq. (1.99) is manifestly written in covariant form, with either side of the equation a Lorentz scalar. In other words, the continuity equation of electrodynamics is Lorentz invariant, even though the individual terms (∂_μ and J^μ) transform as four-vectors.

It is an instructive exercise to show that the continuity equation of electrodynamics is not Galilei invariant. This can be achieved, for example, by using the Galilei boost transformation in Eqs. (5) and the elucidations from section 0.2.

1.4.3 Maxwell's Equations

We begin by defining a four-vector of the electromagnetic potential:

$$\{A^\mu\} = \left\{ \begin{array}{c} A^0 \\ A^k \end{array} \right\} = \left\{ \begin{array}{c} V \\ A_k \end{array} \right\} \qquad \forall k \in \{1, \ldots, 3\} \tag{1.100}$$

where V is the usual scalar potential and $\mathbf{A}$ is the vector potential[17].

Further, we require what is called the **tensor of the electromagnetic field**

$$F^{\mu\nu} := \partial^\mu A^\nu - \partial^\nu A^\mu \tag{1.101}$$

F is here defined as a rank-2 four-tensor with two contravariant indices[18]. Evidently, the left- and the right-hand side of Eq. (1.101) transform in the same manner under Lorentz transformation, but of course F is not a Lorentz scalar.

The elements of the field tensor shall be determined for two examples.

$$\begin{aligned} F^{01} &= \partial^0 A^1 - \partial^1 A^0 = \frac{1}{c}\frac{\partial}{\partial t} A_x - \frac{\partial}{\partial x_1} V \\ &= \frac{\partial}{\partial x^1} V + \frac{1}{c}\frac{\partial}{\partial t} A_x = -\left(-\boldsymbol{\nabla} V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \right)_x \\ &= -E_x \end{aligned}$$

using a combination of the structure equations of electrodynamics

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\ \boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= \mathbf{0} \end{aligned}$$

---

[17]There is a subtlety about notation here: The four-vector is written A as well as the three-vector in non-relativistic notation. Thus, A_k on the right-hand side of Eq. (1.100) are NOT covariant components of A but just the usual cartesian components of $\mathbf{A}$ which transform like contravariant vector elements.

[18]A note on the relationship between $F_{\mu\nu}$ and $F^{\mu\nu}$.....

in the last step. So the structure equations are already contained in the EM field tensor. As a second example consider

$$\begin{aligned} F^{12} &= \partial^1 A^2 - \partial^2 A^1 = \frac{\partial}{\partial x_1} A_y - \frac{\partial}{\partial x_2} A_x \\ &= -\frac{\partial}{\partial x^1} A_y + \frac{\partial}{\partial x^2} A_x = -\left(\boldsymbol{\nabla} \times \mathbf{A}\right)_z \\ &= -B_z \end{aligned}$$

and so on for the remaining elements. Obviously, $F^{\mu\mu} = 0$, and so the field tensor takes on the form

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{1.102}$$

We now postulate that the inhomogeneous Maxwell equations (Gauss's and Ampère's law) can be written in a very elegant and compact form:

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu \tag{1.103}$$

This shall be verified by two examples. First, setting $\nu = 0$:

$$\begin{aligned} \partial_0 F^{00} + \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} &= \frac{4\pi}{c} J^0 \\ 0 + \frac{\partial}{\partial x^1} E_x + \frac{\partial}{\partial x^2} E_y + \frac{\partial}{\partial x^3} E_z &= \frac{4\pi}{c} c\rho \\ \boldsymbol{\nabla} \cdot \mathbf{E} &= 4\pi\rho \qquad (1.104) \end{aligned}$$

which is nothing else than Gauss's law in differential form. Now we set $\nu = 1$:

$$\begin{aligned} \partial_0 F^{01} + \partial_1 F^{11} + \partial_2 F^{21} + \partial_3 F^{31} &= \frac{4\pi}{c} J^1 \\ \frac{1}{c}\frac{\partial}{\partial t}(-E_x) + 0 + \frac{\partial}{\partial x^2} B_z + \frac{\partial}{\partial x^3}(-B_y) &= \frac{4\pi}{c} J_x \qquad (1.105) \\ \left(\boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\right)_x &= \frac{4\pi}{c} J_x \qquad (1.106) \end{aligned}$$

which is one cartesian component of Ampère's law.

The power in the formulation of Eq. (1.103) lies in the fact that it is written in a homogeneous way in terms of four-vectors ($\partial_\mu F^{\mu\nu}$ transforms like a four-vector) which makes it Lorentz covariant, *i.e.*, form invariant with respect to Lorentz transformations. This is by no means obvious in the original formulation of Maxwell's equations.

1.4.4 Relativistic Mass and Linear Momentum

We have seen in the preceding section that the equation of motion of classical mechanics has been generalized to Minkowski space and is formulated in terms of four-vectors. In the following subsections we want to investigate the consequences of this generalization. We begin from a space-like component of the relativistic equation of motion (1.84) and rewrite it:

$$
\begin{aligned}
m_0\, b^k &= K^k \\
m_0\, \frac{du^k}{d\tau} &= \gamma\, F_k \\
m_0\, \frac{du^k}{dt} &= F_k \\
\frac{d\,(m_0\, \gamma\, v_k)}{dt} &= F_k
\end{aligned}
$$

where the definition of four-acceleration (1.80), the proper time differential (1.74), and the obtained expression for four-velocity (1.78) have been used. Form equivalence with Eq. (1.87) and dimensional analysis suggest to define **relativistic linear momentum** as

$$p^k := m_0\, \gamma\, v_k. \tag{1.107}$$

So we have established the space-like components of the linear-momentum four-vector. Before completing the four-vector it is instructive to inspect Eq. (1.113) more closely.

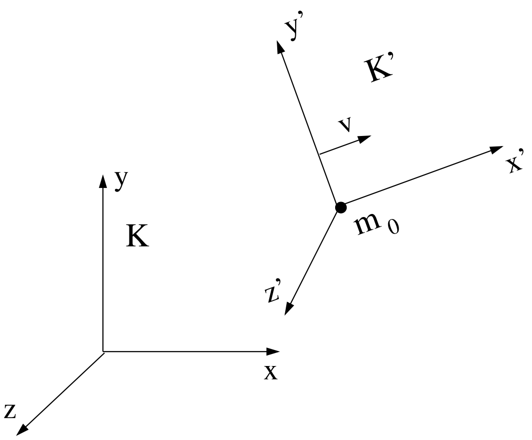
1.4.4.1 Relativistic Mass

In the framework of classical Newtonian mechanics $\mathbf{p} = m\,\mathbf{v}$ where m is the inertial mass of a given particle or body. Since in Eq. (1.113) v_k is the non-relativistic velocity of the particle in frame K and p^k the associated momentum, the implication is that

$$m := \gamma\, m_0 \tag{1.108}$$

should be regarded as the particle's **relativistic inertial mass** in frame K, instead of simply the the rest mass m_0 of the particle. This is a profound difference and means that, since $\gamma = \gamma(v)$ is a function of the velocity of the particle in frame K, so is its mass. The finding is illustrated in Fig. (1.13).

Figure 1.13:



A massive particle with rest mass m_0 in frame K' moves with velocity $\mathbf{v}$ relative to frame K. Its relativistic mass in coordinates of frame K is $\gamma(v)\, m_0$.

Note that rest mass is a Lorentz scalar, i.e., it does not depend on any state of movement. However, in coordinates of frame K, the particle "behaves" as if it had an increased mass, its relativistic or dynamic mass (or observed mass) [19]. It can be anticipated that also the energy of the particle in K should differ from that in frame K', but that is yet to be substantiated.

1.4.4.2 Relativistic Linear Momentum

With these conclusions in mind, the relativistic generalization of linear momentum is straightforward. Since the space-like components are proportional to the dynamic particle mass and its velocity in frame K, the time-like component results by analogy:

$$p^0 = m_0\, \gamma\, c = m_0\, u^0 \tag{1.109}$$

[19] The concept of relativistic mass is not a fundamental requirement. In fact, many authors argue against its introduction since it is sufficient to consider relativistic momentum modified by the γ factor. However, dynamic mass is a useful way of thinking about various situations for instance in the physics of an atom.

Summary for the linear momentum four-vector:

$$\left\{ \begin{array}{c} p^0 \\ p^k \end{array} \right\} = \left\{ \begin{array}{c} mc \\ mv_k \end{array} \right\} \quad \forall k \in \{1, \ldots, 3\} \tag{1.110}$$

As a check for consistency, we take the proper-time derivative of linear momentum,

$$\frac{d}{d\tau}\, p^\mu = m_0\, \frac{d}{d\tau}\, u^\mu = m_0\, b^\mu = K^\mu \tag{1.111}$$

where the relativistic equation of motion (1.84) has been used. We thus obtain a law analogous in form to its non-relativistic counterpart.

1.4.5 Relativistic Energy

We now start from the time-like component of the relativistic fundamental law of dynamics (1.84) and obtain

$$\begin{aligned} m_0\, b^0 &= K^0 \\ m_0\, \frac{du^0}{d\tau} &= \frac{\gamma}{c}\, \mathbf{F}\cdot\mathbf{v} \\ m_0\, \gamma\, \frac{du^0}{dt} &= \frac{\gamma}{c}\, \mathbf{F}\cdot\mathbf{v} \\ \frac{d}{dt}\left(m_0\, \gamma(v)\, c^2\right) &= \mathbf{F}\cdot\mathbf{v} \end{aligned} \tag{1.112}$$

where the expression for the proper time differential (1.74) and the time-like component of four-velocity (1.78) have been used.

Now, since work is $W = \int \mathbf{F}\cdot\mathbf{dx} = \int \mathbf{F}\cdot\mathbf{v}dt$ the right-hand side of Eq. (1.112) corresponds to an amount of work per unit of time. This is equivalent to the temporal change of the quantity in parentheses, which has physical dimensions of energy. We will define this term as **relativistic energy**,

$$E := m_0\, \gamma\, c^2. \tag{1.113}$$

Using the expression for relativistic inertial mass, Eq. (1.108), the relativistic energy can also be written in its (publicly) famous form:

$$\boxed{E = m\, c^2 \qquad (1.114)}$$

The physical meaning of this equation is that every energy corresponds to a mass and every mass corresponds to an energy, with the Lorentz scalar c^2 being the conversion factor.

The most startling consequence of this expression is the special case for a particle at rest with respect to frame K. Then, $||\mathbf{v}|| = 0$ and so $\gamma(||\mathbf{v}||) = 1$ and therefore $m = m_0$. In that case,

$$E_0 = m_0\, c^2 \tag{1.115}$$

and we find that the rest mass of a body corresponds to an energy! This implies that energy should be convertible into rest mass and vice versa[20].

It is very important to analyze the expression Eq. (1.113) before taking any further steps. m_0 and c are Lorentz scalars, but the Lorentz factor γ is a function of velocity. We Taylor expand the Lorentz factor about $v_0 = 0$, resulting in

$$\gamma(v) = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{d^n \gamma(v)}{dv^n} \right|_{v=v_0} (v - v_0)^n \tag{1.116}$$

$$= 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \mathcal{O}\left(\left(\frac{v}{c}\right)^6\right) \tag{1.117}$$

We again see that in the non-relativistic limit, $c \to \infty$, the Lorentz factor becomes 1. The above representation as a Taylor expansion is often useful in order to represent leading relativistic effects by truncating the expansion at some appropriate order.

Using the expansion Eq. (1.117) in Eq. (1.113) one obtains

$$E = m_0\, c^2 + \frac{1}{2} m_0 v^2 + \frac{3}{8} m_0 v^2 \frac{v^2}{c^2} + \mathcal{O}\left(\left(\frac{v}{c}\right)^4\right) \tag{1.118}$$

[20] When formulating this equation, Einstein regarded it true from pure formal aesthetics, but considered believing in its veracity in practice as an "act of faith". It had to be confirmed by experiment, which happened in the decades to come.

In this form relativistic energy can be straightforwardly analyzed:

$\frac{1}{2}\,m_0 v^2$ Beginning with the known term, this represents the kinetic energy of a body with inertial mass $m = m_0$ as in non-relativistic classical mechanics.

$m_0\,c^2$ As a consequence, this term is of relativistic origin. It is evidently a Lorentz scalar, and it relates to an energy of the body independent of kinematics. It is, therefore, called the **rest energy** of the particle.

$\frac{3}{8}\,m_0 v^2\,\frac{v^2}{c^2}$ Since this contribution vanishes in the non-relativistic limit, it is also of relativistic origin and represents the leading **relativistic correction to the particle's kinetic energy**.

$\mathcal{O}\left(\left(\frac{v}{c}\right)^4\right)$ Consequently, all following terms are also relativistic corrections to the particle's kinetic energy.

1.4.6 Relativistic Energy-Momentum Relation

We will now establish an equivalent to the energy-momentum relation from non-relativistic mechanics which reads

$$T = \frac{\left(\mathbf{p}^N\right)^2}{2m_0} \quad \text{with } \mathbf{p}^N = m_0\,\mathbf{v}. \tag{1.119}$$

Stepping back for a moment, we realize that we can identify one relationship between relativistic momentum and relativistic energy right away. From Eqs. (1.110) and (1.114) it follows that the time-like component of the relativistic momentum four-vector is

$$p^0 = mc = \frac{E}{c} \tag{1.120}$$

which means that relativistic energy appears on the time-like component of the linear momentum four-vector, such that we can recast it in another (equivalent form):

$$\{p^\mu\} = \left\{ \begin{array}{c} \frac{E}{c} \\ p^k \end{array} \right\} \qquad \forall k \in \{1, \ldots, 3\} \tag{1.121}$$

The derivation of the relativistic energy-momentum relation is then just a formal exercise. We start from two equivalent forms of the momentum Lorentz scalar $p^2 = p^\mu p_\mu = p^0 p_0 + \sum_k p^k p_k$:

$$\begin{aligned} p^2 &= \frac{E^2}{c^2} - \mathbf{p}^2 \\ p^2 &= m_0^2\,\gamma^2\,c^2 - m_0^2\,\gamma^2\,\mathbf{v}^2 \end{aligned} \tag{1.122}$$

where the second relation follows from Eqs. (1.107) and (1.110). The first relation can be rewritten as

$$E^2 = c^2\left(p^2 + \mathbf{p}^2\right) \tag{1.123}$$

and inserting the second relation into it yields

$$\begin{aligned} E^2 &= \mathbf{p}^2\,c^2 + m_0^2\,c^2\left(\gamma^2\,c^2 - \gamma^2\,\mathbf{v}^2\right) \\ &= \mathbf{p}^2\,c^2 + m_0^2\,c^2\left(\frac{c^2}{1-\frac{\mathbf{v}^2}{c^2}} - \frac{\mathbf{v}^2}{1-\frac{\mathbf{v}^2}{c^2}}\right) \\ &= \mathbf{p}^2\,c^2 + m_0^2\,c^4\left(\frac{c^2}{c^2-\mathbf{v}^2} - \frac{\mathbf{v}^2}{c^2-\mathbf{v}^2}\right) \end{aligned}$$

and so we obtain

$$E^2 = \mathbf{p}^2\,c^2 + m_0^2\,c^4. \tag{1.124}$$

Taking the positive square root gives

$$\boxed{E = \sqrt{\mathbf{p}^2\,c^2 + m_0^2\,c^4} \quad (1.125)}$$

which is known as the **relativistic energy-momentum relation**. Note that the first term under the square root is the square of linear *three*-momentum, not to be confused[21] with the scalar product of four-momentum in Eq. (1.122).

At the time of its first appearance, there was no dispute about taking into account the positive square root only, although formally the negative square root could also be permissible. After all, the notion of *negative energy* seems queer. This point became a remarkable twist in the history of physics and will be picked up again later on in the context of relativistic quantum mechanics.

A number of interesting conclusions can be drawn from Eq. (1.125) when considering various possible cases, for which it is equally valid. The first distinction concerns the body's rest mass, m_0.

1.4.6.1 Particles with zero rest mass

As a first observation, we note that this case is particular to relativistic theory. The notion of a particle with zero mass makes no sense in non-relativistic mechanics. Due to the intimate relationship between energy and mass in Eq. (1.114), however, we must take this possibility seriously here.

Omitting the rest-mass term from Eq. (1.125) yields

$$E = ||\mathbf{p}||\, c. \tag{1.126}$$

However, we know that relativistic three-momentum is

$$\mathbf{p} = m_0\, \gamma(\mathbf{v})\, \mathbf{v}$$

so the energy of the "particle" seems to be zero, except if the particle is allowed to travel at the speed of light in frame K, in which case the γ factor tends to infinity! The problem obviously remains not

[21]Some texts use unclear notation on this point.

fully resolved in classical relativistic mechanics, but a first glance at **quantum mechanics** in this context reveals an interesting connection:

Inserting de Broglie's relation ($p = \frac{h}{\lambda}$) and Planck-Einstein's relation ($E = h\nu$) which are valid for de Broglie matter waves, where h is Planck's constant, into Eq. (1.126) we obtain

$$\begin{aligned} h\nu &= \frac{h}{\lambda}\, c \\ \nu &= \frac{c}{\lambda} \end{aligned}$$

which is the well-known relationship between frequency and wavelength for waves propagating according to Maxwell's equations[22].

We conclude that the theory of **massless particles** is **necessarily a relativistic quantum theory** which will be developed in later sections.

1.4.6.2 Massive Particles

For massive particles we can discuss two limiting cases:

$v << c$. From Eq. (1.118) in this approximation it follows that

$$E \approx m_0\, c^2 + \frac{\left(\mathbf{p}^N\right)^2}{2m_0} \qquad \text{for } m_0 \neq 0 \tag{1.127}$$

$v \approx c$. According to Eq. (1.110) for the linear relativistic three-momentum we can write

$$\mathbf{p}^2\, c^2 = m_0^2 \gamma^2\, \mathbf{v}^2 c^2 = m_0^2 c^4\, \frac{\mathbf{v}^2}{c^2 - \mathbf{v}^2} \tag{1.128}$$

[22] Note that trying to make the same argument based on non-relativistic momentum $p = mv$ does not lead to a consistent theory. In that case, since $\lambda = \frac{h}{mc}$, supposing propagation at the speed of light, the wavelength for a particle whose mass tends to zero becomes infinite which is in contradiction with observation. In other words, non-relativistic quantum mechanics "works" for massive particles at lower velocities.

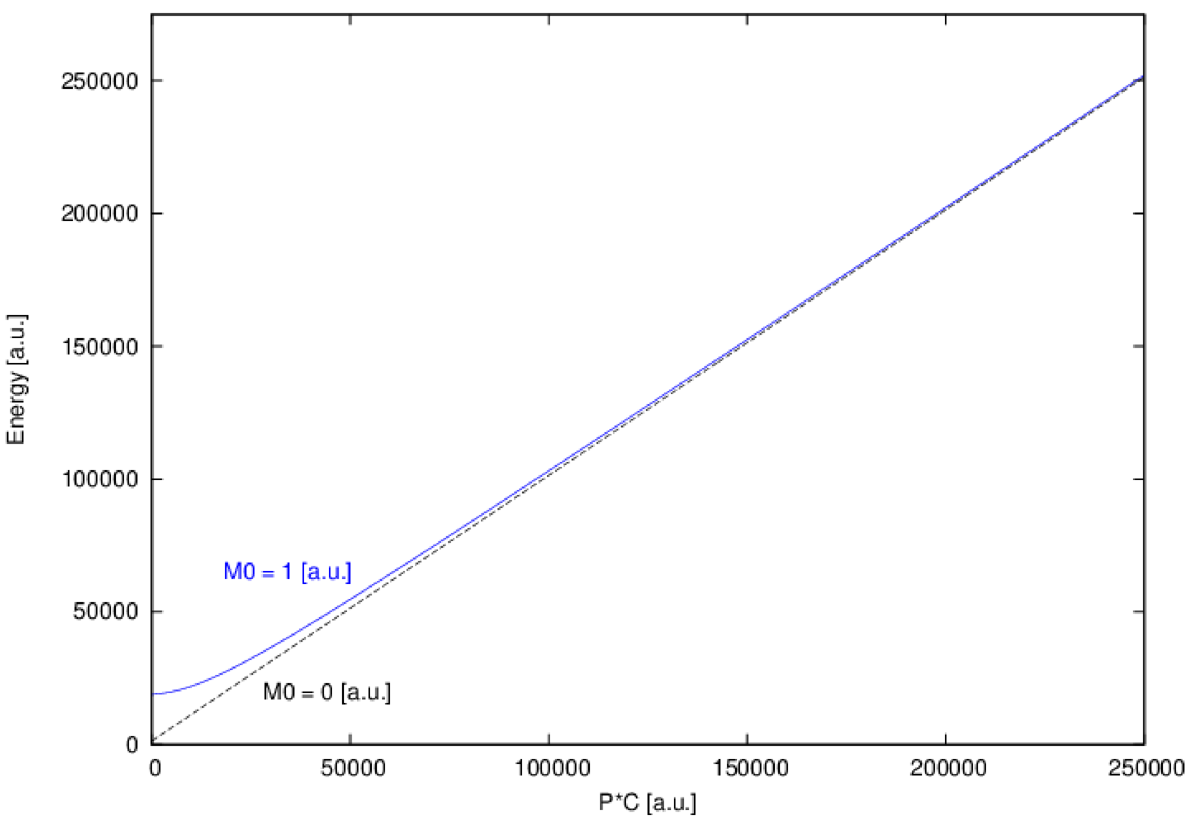
Now since $\frac{\mathbf{v}^2}{c^2-\mathbf{v}^2} >> 1$ with the applied condition it follows that here $\mathbf{p}^2\, c^2 >> m_0^2 c^4$ and it can be approximated

$$E \approx ||\mathbf{p}||\, c \qquad \text{for } m_0 \neq 0 \tag{1.129}$$

in this limit. As a consequence, for very large relative velocities the rest energy becomes negligible as a contribution to the total relativistic energy.

Summary. Relativistic energy as a function of linear momentum for vanishing (e.g. for the photon) and non-vanishing rest mass is depicted in Fig. (1.14).

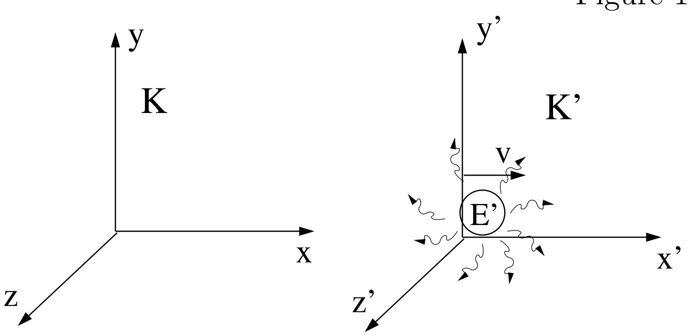
Figure 1.14:



1.4.7 Relationship Energy-Mass: Mass Defect

In order to deepen the understanding of Eqs. (1.114) and (1.125) we will consider the following thought experiment. Be there a system at rest with respect to a frame K' that moves with velocity v relative to a laboratory frame K, Fig. (1.15).

Figure 1.15:



During a short time span $\Delta t'$ the system emits radiation of energy E' in K' symmetrically such that its total momentum in K' does not change, *i.e.*, it remains at rest in K'.

Such an emission process may, for example, occur in the formation of an atomic nucleus from its constituent nucleons (protons and neutrons). The potential energy (attractive strong interaction between nucleons) is released in form of radiation.

The momentum four-vector for this scenario takes on the following form in K and K', respectively:

Momentum four-vector in K: Momentum four-vector in K':

$$\{p^\mu\} = \left(\frac{E}{c}, p^1 \equiv p, 0, 0\right) = \left(\frac{E}{c}, \frac{v}{c^2}\,E, 0, 0\right) \qquad \{p^{\mu\prime}\} = \left(\frac{E'_{\text{rad}}}{c}, 0, 0, 0\right)$$

For the manipulation in K we have used $\mathbf{p} = m\mathbf{v} = \mathbf{v}\,\frac{mc^2}{c^2} = \frac{\mathbf{v}}{c^2}\,E$ with use of Eq. (1.114).

Of course, the momentum four-vector $\{p^{\mu\prime}\}$ has to be related with $\{p^\mu\}$ through a Lorentz transformation. In the present case we want to transform from K' to K, so we use $\mathbf{\Lambda}^{-1}(v) = \mathbf{\Lambda}(-v)$ with respect to

Eq. (1.51). The transformation of momentum for the boost then reads

$$\mathbf{\Lambda}(-v) \begin{pmatrix} p^{0'}_{\text{rad}} \\ p^{1'}_{\text{rad}} \end{pmatrix} = \begin{pmatrix} p^{0}_{\text{rad}} \\ p^{1}_{\text{rad}} \end{pmatrix}$$
$$\begin{pmatrix} \gamma & \frac{v}{c}\gamma \\ \frac{v}{c}\gamma & \gamma \end{pmatrix} \begin{pmatrix} \frac{E'_{\text{rad}}}{c} \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma \frac{E'_{\text{rad}}}{c} \\ \frac{v}{c}\gamma \frac{E'_{\text{rad}}}{c} \end{pmatrix} \tag{1.130}$$

The resulting four-vector has to be identical with the original formulation of the momentum four-vector in frame K. Let us first inspect the first space-like component (first component of three-momentum). We find

$$p_{\text{rad}} = \frac{v}{c}\gamma \frac{E'_{\text{rad}}}{c} \tag{1.131}$$

which is the momentum (in K) corresponding to the energy of the radiation pulse in K'. However, the system did not change its momentum in K', due to the assumed symmetrical emission. This implies that its velocity relative to the laboratory has also not changed. What are the consequences?

At this point, it is imperative to exploit principles of symmetry. The system of our thought experiment is **isolated**. Therefore, it is **invariant to a spatial translation** and so its **total momentum in a given frame is conserved**.

The basic form of momentum in accord with Eq. (1.110) is

$$p = m_0 \, \gamma \, v \tag{1.132}$$

for the component of interest, where m_0 here is the rest mass of the system. This quantity has to be conserved due to symmetry. But Eq. (1.131) forces us to consider the momentum of the radiation pulse that is non-zero in the balance of momentum in K. And since relative velocity does not change due to the pulse, the only way to compensate is by a loss of rest mass of the system due to the emission of radiation!

Formally, momentum conservation in K is thus written as

$$\begin{aligned} p_{\text{before}} &= p_{\text{after}} + p_{\text{rad}} \\ m_{0\text{before}}\,\gamma\, v &= m_{0\text{after}}\,\gamma\, v + \frac{v}{c}\,\gamma\,\frac{E'_{\text{rad}}}{c} \end{aligned} \tag{1.133}$$

from which it follows that

$$m_{0\text{before}} = m_{0\text{after}} + \frac{E'_{\text{rad}}}{c^2}. \tag{1.134}$$

The energy of the radiation pulse divided by the square of the speed of light is the **rest mass lost** by the system due to the emission of the radiation pulse of energy E'. The system has, therefore, suffered a **mass defect**.

Typically, lighter nuclei than iron represent this situation. Energy has to be invested in order to separate them into nucleons. This energy then shows in the form of the increased mass of the sum of the nucleons compared with the rest mass of the nucleus. Heavy (and superheavy) nuclei tend to release energy when they separate into constituents (nuclear fission).