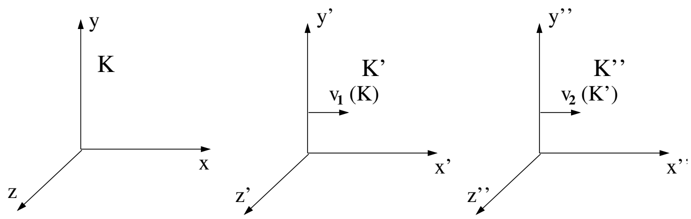


Inertial frames K, K' and K" with axes aligned. Origins coincide at $t = t' = t'' = 0$.

original form. The two Lorentz boosts can thus be written as

$$\begin{array}{cc} x' = \gamma_1(x - v_1 t) & x'' = \gamma_2(x' - v_2 t') \\ t' = \gamma_1(t - x\frac{v_1}{c^2}) & t'' = \gamma_2(t' - x'\frac{v_2}{c^2}) \\ \mathrm{K} \to \mathrm{K'} & \mathrm{K'} \to \mathrm{K''} \end{array} \tag{1.34}$$

where individual Lorentz factors $\gamma_j(v_j) = \frac{1}{\sqrt{1-\frac{v_j^2}{c^2}}}$ have been introduced. Inserting the first transformation into the second corresponds to transforming coordinates from K → K" and results in

$$\begin{aligned} x'' &= \gamma_1\gamma_2\, x \left(1 + \frac{v_1 v_2}{c^2}\right) - \gamma_1\gamma_2\, t(v_1 + v_2) \\ t'' &= \gamma_1\gamma_2\, t \left(1 + \frac{v_1 v_2}{c^2}\right) - \gamma_1\gamma_2\, \frac{x}{c^2}(v_1 + v_2) \end{aligned} \tag{1.35}$$

after convenient regrouping of terms[11]. However, the Lorentz transformation relating K → K" can also be written in a general form:

$$\begin{aligned} x'' &= \gamma_3(x - v_3 t) \\ t'' &= \gamma_3 \left(t - x\frac{v_3}{c^2}\right) \end{aligned} \tag{1.36}$$

where v_3 is the relative velocity of these two frames. Comparing coefficients of x and t in Eqs. (1.35) and (1.36) results in (two times) the following conditions:

$$\gamma_3 = \gamma_1\gamma_2 \left(1 + \frac{v_1 v_2}{c^2}\right) \tag{1.37}$$

$$\gamma_3\, v_3 = \gamma_1\gamma_2(v_1 + v_2) \tag{1.38}$$

[11] We could also obtain these two equations by calculating the matrix product $\tilde{\mathbf{L}}(v_2)\tilde{\mathbf{L}}(v_1)$ and then acting with the new matrix onto the SpaceTime coordinates $\begin{pmatrix} x \\ t \end{pmatrix}$.

Inserting the first into the second of these conditions directly gives

$$v_3\,\gamma_1\gamma_2\left(1+\frac{v_1v_2}{c^2}\right)=\gamma_1\gamma_2(v_1+v_2)$$

$$\boxed{v_3=\frac{v_1+v_2}{1+\frac{v_1v_2}{c^2}}} \tag{1.39}$$

the **addition theorem for velocities** in special relativity for relative movement along aligned coordinate axes. Based on it a satisfactory explanation and interpretation of the experiments carried out by Fizeau, Michelson and Morley and others is possible.

1.1.3.2 Corollaries

A number of interesting consequences of the addition theorem for velocities shall be discussed at this point. Considering the properties of the Lorentz factor we have seen that $v_j > c$ is generally unacceptable. So we *assert* that $v_j \leq c$ is always true and investigate its consequences.

- $v_3 \leq c$.
  Proof. $\lim\limits_{v_1,v_2\to c} v_3 = \frac{2c}{2} = c$. And also $\lim\limits_{v_2\to c} = \frac{v_1+c}{1+\frac{v_1}{c}} = \frac{c(v_1+c)}{v_1+c} = c$.

- $\lim\limits_{c\to\infty} v_3 = v_1 + v_2$
  which establishes the addition theorem for velocities in the non-relativistic limit.

1.1.4 Properties of the Lorentz Transformation

The first and foremost question concerns the invariance of physical laws under coordinate transformations. For example, Newton's second law is invariant under spatial rotations. So what does the obtained LT represent?

We may write the one-dimensional Lorentz boost given in Eq. (1.28) also as a matrix equation,

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{v}{c^2}\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \tag{1.40}$$

where the vector space we will henceforth call **SpaceTime** has been introduced. It is the natural generalization of the usual three-dimensional coordinate space $\mathbb{R}^3$ to include the time coordinate, so Minkowski SpaceTime is defined as $\mathbb{R}^4$.

Since the y and z coordinates are left unaffected by the boost, we may for the present case simplify the matrix equation to two dimensions:

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \gamma & -v\gamma \\ -\frac{v}{c^2}\gamma & \gamma \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} = \tilde{\mathbf{L}}\,\mathbf{x} \tag{1.41}$$

We find that

- $det(\tilde{\mathbf{L}}) = 1$
- $\mathbf{x}'^2 \neq \mathbf{x}^2$; non-conservation of the scalar product, naïvely defined as in usual $\mathbb{R}^2$
- $\tilde{\mathbf{L}}^T\,\tilde{\mathbf{L}} \neq \mathbf{1}$ and so $\tilde{\mathbf{L}}^T \neq \tilde{\mathbf{L}}^{-1}$.

This means that $\tilde{\mathbf{L}}$ is not an orthogonal matrix. So clearly $\tilde{\mathbf{L}}$ does not represent a rotation in Minkowski SpaceTime. Note also that the physical dimensions of the SpaceTime vector $\begin{pmatrix} x \\ t \end{pmatrix}$ are not homogeneous.

In order to obtain an orthogonal transformation in SpaceTime Minkowski introduced a trick: He defined one SpaceTime vector component as *imaginary*, *i.e.* we have the SpaceTime coordinates $\{ict, x\}$ instead of

$\{x, t\}$. Then the Lorentz transformation becomes

$$\begin{pmatrix} x' \\ \imath ct' \end{pmatrix} = \begin{pmatrix} \gamma & \imath\frac{v}{c}\gamma \\ -\imath\frac{v}{c}\gamma & \gamma \end{pmatrix} \begin{pmatrix} x \\ \imath ct \end{pmatrix} = \mathbf{L}\,\mathbf{x} \tag{1.42}$$

The reader can easily verify that this version of the Lorentz boost is identical to the original one above, Eq. (1.41). However, the Space-Time vector components are now physically homogeneous. In addition, the following properties of the transformation matrix with respect to Minkowski coordinates can be shown straightforwardly:

- $det(\mathbf{L}) = 1$
- $\mathbf{x}'^2 = \mathbf{x}^2$; conservation of the scalar product[12]
- $\mathbf{L}^T = \mathbf{L}^{-1}$; orthogonality
- $v \longrightarrow -v \Rightarrow \mathbf{L} \longrightarrow \mathbf{L}^{-1}$; inverse transformation property

The conservation of the scalar product can also be written as

$$\mathbf{x}'^2 = \mathbf{x}^2 \tag{1.43}$$

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = x^2 + y^2 + z^2 - c^2t^2 \tag{1.44}$$

by taking all three spatial components. This finding has a direct physical interpretation. If at $t = 0$ a light pulse is emitted from the origin of inertial frame K, then its radial position at time t is

$$r = \sqrt{x^2 + y^2 + z^2} = ct \tag{1.45}$$

$$\Rightarrow x^2 + y^2 + z^2 - c^2t^2 = 0. \tag{1.46}$$

In coordinates of K', where the origins coincide at $t = t' = 0$, we infer from the constancy of the speed of light in all frames (Postulate 4):

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = 0 \tag{1.47}$$

[12]The scalar product in Minkowski space is here (!) defined with the usual Euclidian metric tensor, so here $\mathbb{1}_2$.

Equating the two above expressions reproduces the conservation of the scalar product under Lorentz transformation.

In fact, the Lorentz invariant $x^2+y^2+z^2-c^2t^2$ comprises a so-called **Lorentz scalar**. We will come back to a more general discussion of Lorentz scalars in a later section.

Note that the scalar product can also be written as $x^2+y^2+z^2+(\imath ct)^2$ which comprises the natural form of a scalar product in a four-dimensional space.

1.1.5 Minkowski Metric

The modern standard representation of the Lorentz transformation is different from both the ones we have established in subsection 1.1.4. Let us review the scalar product of the SpaceTime vector $\begin{pmatrix} x' \\ \imath ct' \end{pmatrix}$ with itself (Eq. (1.42)):

$$\begin{aligned} \begin{pmatrix} x' & \imath ct' \end{pmatrix} \begin{pmatrix} x' \\ \imath ct' \end{pmatrix} &= \begin{pmatrix} x & \imath ct \end{pmatrix} \begin{pmatrix} \gamma & -\imath \frac{v}{c}\gamma \\ \imath \frac{v}{c}\gamma & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \imath \frac{v}{c}\gamma \\ -\imath \frac{v}{c}\gamma & \gamma \end{pmatrix} \begin{pmatrix} x \\ \imath ct \end{pmatrix} \\ &= \begin{pmatrix} x & \imath ct \end{pmatrix} \begin{pmatrix} \gamma^2 - \frac{v^2}{c^2}\gamma^2 & 0 \\ 0 & -\frac{v^2}{c^2}\gamma^2 + \gamma^2 \end{pmatrix} \begin{pmatrix} x \\ \imath ct \end{pmatrix} \\ &= \begin{pmatrix} x & \imath ct \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ \imath ct \end{pmatrix} = x^2 - c^2t^2 \end{aligned} \tag{1.48}$$

where we have used the usual 3-dimensional Euclidean metric for the scalar product, a unit matrix[13].

But it is possible to write the same scalar product for **real-valued coordinate axes** by changing the form of the metric[14].

The Euclidean metric in two-dimensional flat space is defined as

$$\mathbf{g} = \begin{pmatrix} \mathbf{e}_x \cdot \mathbf{e}_x & \mathbf{e}_x \cdot \mathbf{e}_y \\ \mathbf{e}_y \cdot \mathbf{e}_x & \mathbf{e}_y \cdot \mathbf{e}_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.49}$$

[13] in the present case 2-dimensional.

[14] Likewise, the scalar product over two vectors in a vector space when changing from orthogonal to non-orthogonal axes can be conserved by introducing an accompanying change of the metric.

Now a coordinate transformation is characterized by the corresponding Jacobian matrix $\mathbf{J}$[15]. Suppose we define the coordinate transformation as follows:

$$\begin{aligned} u' &= u \\ v' &= \imath v \end{aligned} \tag{1.53}$$

Then we get the Jacobian of the transformation as

$$\mathbf{J} = \begin{pmatrix} \frac{\partial u'}{\partial u} & \frac{\partial u'}{\partial v} \\ \frac{\partial v'}{\partial u} & \frac{\partial v'}{\partial v} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \imath \end{pmatrix} \tag{1.54}$$

The conservation of the scalar product under coordinate transformation is achieved by transforming the metric, since

$$\begin{pmatrix} u' & v' \end{pmatrix} \mathbf{g} \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} u & v \end{pmatrix} \mathbf{J}^T \mathbf{g}\,\mathbf{J} \begin{pmatrix} u \\ v \end{pmatrix} \tag{1.55}$$

and so we can define a new metric as

$$\mathbf{g}' = \mathbf{J}^T \mathbf{g}\,\mathbf{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.56}$$

for the scalar product. And so we have

$$\begin{pmatrix} u' & v' \end{pmatrix} \mathbf{g} \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} u & v \end{pmatrix} \mathbf{g}' \begin{pmatrix} u \\ v \end{pmatrix} \tag{1.57}$$

which means that the scalar product is conserved under coordinate transformation if the metric tensor is transformed accordingly.

In special relativity coordinates this means

$$\begin{pmatrix} x & \imath ct \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ \imath ct \end{pmatrix} = \begin{pmatrix} x & ct \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}$$

where the new non-unity metric for the scalar product has been introduced. Moreover, as we have seen earlier, for the conservation of the scalar product $x^2 - c^2t^2 = x'^2 - c^2t'^2$ (Eq. (1.44)) the global sign is

[15] in the following way: The total differentials for the set of coordinates can be written as

$$du' = \frac{\partial u'}{\partial u} du + \frac{\partial u'}{\partial v} dv \tag{1.50}$$

$$dv' = \frac{\partial v'}{\partial u} du + \frac{\partial v'}{\partial v} dv \tag{1.51}$$

where $du' = u'_1 - u'_2$ is an infinitesimal interval along u'. $u'_2 = 0$ is just a special case of this. Arranging this in matrix form makes the Jacobian matrix appear:

$$\begin{pmatrix} du' \\ dv' \end{pmatrix} = \begin{pmatrix} \frac{\partial u'}{\partial u} & \frac{\partial u'}{\partial v} \\ \frac{\partial v'}{\partial u} & \frac{\partial v'}{\partial v} \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix} \tag{1.52}$$

The rest follows from there. See texts on metric tensors for more information.

irrelevant; what matters is only the relative sign between spatial and time coordinates. So we are free to choose the metric according to

$$\begin{pmatrix} ct & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} = c^2t^2 - x^2 \tag{1.58}$$

where the "minus" sign is on the spatial coordinate instead. This metric has become the modern standard[16]. We will call

$$\mathbf{g} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.59}$$

modern Minkowski metric, for the four dimensions of SpaceTime, and use it from here onward. Moreover, in the new basis of SpaceTime (where the time coordinate also becomes the first coordinate) the Lorentz transformation matrix changes. The reader can easily verify the equivalence of the two representations:

$$\begin{pmatrix} \imath ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\imath\frac{v}{c}\gamma \\ \imath\frac{v}{c}\gamma & \gamma \end{pmatrix} \begin{pmatrix} \imath ct \\ x \end{pmatrix} \quad \text{with complex axes} \tag{1.60}$$

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{v}{c}\gamma \\ -\frac{v}{c}\gamma & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \quad \text{with real axes} \tag{1.61}$$

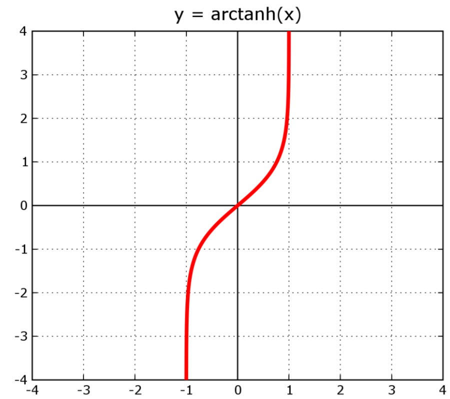
From now on we use the designation $\mathbf{\Lambda}(v) := \begin{pmatrix} \gamma & -\frac{v}{c}\gamma \\ -\frac{v}{c}\gamma & \gamma \end{pmatrix}$ for Lorentz boosts.

1.1.6 Lorentz transformation in Terms of Rapidity

There is a convenient way of expressing the Lorentz boost which is useful in the context of combined Lorentz transformations and the **Lorentz group**. If we define the "rapidity"

[16] It has been the standard for at least 50 years. A landmark text that still uses the old unit metric is Bethe and Salpeter, "Quantum Mechanics of One- and Two-Electron Atoms".

$$\Phi_x := \operatorname{arctanh}\left(\frac{v}{c}\right) \qquad (1.62)$$



then it follows that

$$\begin{aligned}
\frac{v}{c} &= \tanh\Phi_x \\
\frac{v^2}{c^2} &= \frac{\cosh^2\Phi_x - 1}{\cosh^2\Phi_x} \\
\gamma &= \cosh\Phi_x \\
\frac{v}{c}\gamma &= \sinh\Phi_x
\end{aligned} \qquad (1.63)$$

and so we obtain for the modern representation of the Lorentz transformation

$$\begin{pmatrix} \gamma & -\frac{v}{c}\gamma \\ -\frac{v}{c}\gamma & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \cosh\Phi_x & -\sinh\Phi_x \\ -\sinh\Phi_x & \cosh\Phi_x \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \qquad (1.64)$$

It is then a straightforward exercise to prove the following identity:

$$\begin{aligned}
\mathbf{\Lambda}(\Phi_{x_1})\mathbf{\Lambda}(\Phi_{x_2}) &= \mathbf{\Lambda}(\Phi_x) \\
&\text{with} \\
\Phi_{x_1} + \Phi_{x_2} &= \Phi_x
\end{aligned} \qquad (1.65)$$

Eq. (1.65) shows that the double boost occurring, e.g., in the derivation of the velocity addition theorem can — using the rapidity parameter — be written conveniently as a new Lorentz transformation where the new rapidity Φ_x simply is the sum of the two original rapidities. This resembles the formal situation for the addition of velocities in Newtonian mechanics using the Galilei transformation.

1.2 Four-Vectors in SpaceTime

1.2.1 Inverse Lorentz Transformation

If we want to represent the Lorentz boost from frame K' into frame K, we need the inverse Lorentz transformation matrix. It can be derived straightforwardly from Eq. (1.61) to be

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \gamma & \frac{v}{c}\gamma \\ \frac{v}{c}\gamma & \gamma \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}. \tag{1.66}$$

Therefore, $\mathbf{\Lambda}^{-1}(v) = \mathbf{\Lambda}(-v)$. This result is physically intuitive, since the origins of K and K' propagate in opposite directions relative to the respective other frame. It is easily verified that $\mathbf{\Lambda}^{-1}(v)\,\mathbf{\Lambda}(v) = \mathbf{\Lambda}(v)\,\mathbf{\Lambda}^{-1}(v) = \mathbb{1}_4$ for the four-dimensional case. We are now in the position to introduce general four-vectors in Minkowski SpaceTime.

1.2.2 Four-Vectors (Co- and Contravariant)

1.2.2.1 Position four-vector

We have already given the time coordinate in SpaceTime physical dimension of length (through the multiplication with the speed of light), and so, with $x_0 = ct$, it is suggested to introduce a **position four-vector** as

$$x := \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \text{With components}\,\{x_\mu\} \tag{1.67}$$

where the x_0 is the **time-like** component and $x_k, k \in \{1, \ldots, 3\}$ are the cartesian **space-like** components[17]. Then we can write the Lorentz boost also as

$$x' = \mathbf{\Lambda}_{\mathrm{K}\to\mathrm{K'}}\, x \tag{1.68}$$

[17] In the present case of the position vector, the time-like component is actually time itself and the space-like components represent space itself.

or, using component notation,

$$x'_\nu = \sum_{\mu=0}^{3} \Lambda_{\nu\mu}\, x_\mu \qquad \forall \nu \in \{0, \ldots, 3\} \tag{1.69}$$

This is the Minkowski-space equivalent of the transformation law of a vector, a three-tensor of rank 1, in real space! Here we transform a four-vector in Minkowski space.

From now on, we will use **Einstein summation convention** which is defined as a sum in Minkowski space (or coordinate space) over duplicate indices in any given term. Then,

$$x'_\nu = \Lambda_{\nu\mu}\, x_\mu. \tag{1.70}$$

From the above it immediately follows that

$$\frac{\partial x'_\nu}{\partial x_\mu} = \Lambda_{\nu\mu}. \tag{1.71}$$

Conversely, from Eq. (1.70), we have

$$\begin{aligned} \left(\Lambda^{-1}\right)_{\kappa\nu} x'_\nu &= \left(\Lambda^{-1}\right)_{\kappa\nu} \Lambda_{\nu\mu}\, x_\mu \\ &= \delta_{\kappa\mu}\, x_\mu \\ &= x_\kappa \\ \left(\Lambda^{-1}\right)_{\mu\nu} x'_\nu &= x_\mu. \end{aligned} \tag{1.72}$$

and so

$$\frac{\partial x_\mu}{\partial x'_\nu} = \left(\Lambda^{-1}\right)_{\mu\nu}. \tag{1.73}$$

1.2.2.2 General Contra- and Covariant Four-Vectors

Let us now take a first look at fields in special relativity. Be $\varphi(x)$ a scalar differentiable field with x the position four-vector in frame K. We regard the derivative with respect to coordinates in K', *i.e.*,

$$\frac{\partial \varphi(x)}{\partial x'_\mu} = \frac{\partial \varphi(x)}{\partial x_\kappa} \frac{\partial x_\kappa}{\partial x'_\mu} \tag{1.74}$$

since $\varphi\left(x(x')\right)$, where in the second equality the chain rule and Einstein summation have been used. This means that we can write for the differential operator

$$\frac{\partial}{\partial x'_\mu} = \frac{\partial}{\partial x_\kappa}\frac{\partial x_\kappa}{\partial x'_\mu} = \frac{\partial}{\partial x_\kappa}\left(\Lambda^{-1}\right)_{\kappa\mu} \tag{1.75}$$

where we have used Eq. (1.73). If we compare this result with Eq. (1.70), we see a difference: The components of the position four-vector are transformed from K to K' *via* the transformation matrix Λ, but the components of the derivative vector with respect to position transform — also from K to K' — *via* the *inverse* transformation matrix Λ^{-1}! This implies that there are, generally speaking, two types of four-vectors in relativity theory[18]:

1. The **contravariant components** of four-vectors transform from K to K' as to the above Lorentz transformation. They are, by convention, written with upper indices, a^μ.

2. The **covariant components** of four-vectors transform from K to K' as to the inverse Lorentz transformation. They are conversely written with lower indices, b_ν.

Let us now consider the scalar product of a covariant four-vector with a contravariant four-vector in Minkowski SpaceTime. For consistency with general matrix algebra, we will here consider covariant vectors as row vectors, i.e., $x'_\mu = x_\nu\left(\Lambda^{-1}\right)^\nu_{\ \mu}$ and always **sum over repeated upper and lower indices**. Then

$$\begin{aligned} b'_\mu a^{\mu'} &= b_\nu\left(\Lambda^{-1}\right)^\nu_{\ \mu}\Lambda^\mu_{\ \kappa}\,a^\kappa \\ &= b_\nu\,\delta^\nu_\kappa\,a^\kappa && (1.76) \\ &= b_\nu\,a^\nu = b_\mu\,a^\mu. && (1.77) \end{aligned}$$

[18]The reader may argue that this situation actually already exists in non-relativistic physics for position and gradient vectors, for example. This is true, but for a Euklidean metric there is no difference between the contra- and covariant components of a vector!

We find that the scalar product of a covariant with a contravariant vector is invariant under Lorentz transformation. $b_\nu\, a^\nu$ is therefore called a **Lorentz scalar**. It can without difficulties be shown that $a^{\mu'} b'_\mu = a^\nu b_\nu$, and so the scalar product of any contravariant with any covariant four-vector is also a Lorentz scalar.

The result in Eq. (1.77) for general four-vectors will be of utmost importance in the construction of relativistic theories.

1.2.2.3 Relationship Between Contra- and Covariant Four-Vectors

In Eq. (1.58) we had agreed that the scalar product involves the "modern metric", $\{g_{\mu\nu}\}$. It shall by definition have the property $g_{\mu\nu} = g^{\mu\nu}$. From this it directly follows that

$$(gg)^\kappa_\mu = g_{\mu\nu}\, g^{\nu\kappa} = \delta^\kappa_\mu \tag{1.78}$$

where δ^κ_μ is the usual Kronecker delta symbol[19].

Let us now see how contra- and covariant components of a four-vector are related to each other. The expression $b_\nu\, a^\nu$ is a Lorentz scalar. So is $b^\nu\, a_\nu$. Since these two Lorentz scalars are made up of the same four-vectors, they should be identical. We now suppose that

$$b^\nu\, a_\nu = b_\mu\, g^{\mu\nu}\, g_{\nu\kappa}\, a^\kappa \tag{1.79}$$

which can simply be calculated, using Eq. (1.78):

$$b_\mu\, g^{\mu\nu}\, g_{\nu\kappa}\, a^\kappa = b_\mu\, \delta^\mu_\kappa\, a^\kappa = b_\mu\, a^\mu = b_\nu\, a^\nu \tag{1.80}$$

By comparing Eqs. (1.79) and (1.80) we have the following relationships between co- and contravariant indices of a given four-vector:

$$b^\nu \;=\; b_\mu\, g^{\mu\nu} \tag{1.81}$$

$$a_\nu \;=\; g_{\nu\kappa}\, a^\kappa \tag{1.82}$$

[19] In the case of diagonal matrices we do not have to care about which index is the row and which is the column index, so we use this simplified notation.