Special Relativity and Nuclear Physics

(and some Particle Physics)

L3

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Contents

Preface

These lecture notes evolved between the years 2017 and 2022 during the course I was giving to third-year physics students at the University Paul Sabatier. Its spirit is an axiomatic (as opposed to historical) introduction to special relativity, followed by applications in particle and nuclear physics. The Covid crisis in the last two years of the lecturing period – where many courses and exercise sessions had to be given online – helped substantially in completing the LaTeX manuscript which is now fully available.

Some comments on recommended literature:

- **W. Rindler**: Relativity.
  Covers special and general relativity; good and established introductory text.
- **Jackson**: Classical Electrodynamics.
  Lorentz-covariant formulation of electrodynamics
- **D. Griffiths**: Introduction to Elementary Particles.
  Pedagogically brilliant introduction to relativistic kinetics and theoretical particle physics

Chapter 0

Introduction

0.1 Generalities

Welcome to this L3 course on “Relativity and Nuclear Physics”. Let us begin with some general outline and positioning of the matters in the complete framework of modern physics.

The theory of relativity is one of the centrals pillars of modern physics (the other being quantum mechanics). The theory of **Special Relativity** was developed first (1905), and it reconciles Newton’s laws of motion with electrodynamics. Newton’s laws of motion are invariant under Galilean transformations, whereas classical electrodynamics was known not to be invariant under such transformations[1]. Newtonian physics can thus be rewritten in the framework of special relativity which is a first goal of this course. After this modification, Newton’s laws will not be the same anymore, although they will retain an **equivalent structure**. To the contrary, Maxwell’s electrodynamics will remain unchanged, but the equations will be written in a new language that is adapted to the principles of special relativity.

Later in the history of physics (1926), even quantum mechanics underwent a first round of unification with special relativity in the form of the **Dirac equation**. The course will cover this development in

[1] This fact could be demonstrated, but we will take a different route: We will develop the relativistic Einsteinian – different from the Galilean – transformation and show that Maxwell’s equations are invariant under this new transformation.

the second half of the semester. The next steps were taken in the late 1940s and early 1950s when Quantum Electrodynamics and more generally Quantum Field Theory were developed, representing a complete "merger" of special relativity with quantum mechanics. These developments even encompassed two forces beyond the electromagnetic force, the **nuclear "strong" force** and the **"weak" force**. The quantum-field theoretical framework for these three forces is today known as the "Standard Model of Elementary Particles" (SM), developed in the 1960s and 1970s. It was completed in 2012 with the detection of the evasive SM "Higgs boson" at the CERN laboratories.

The (so far) last of Nature's forces, the gravitational force, was incorporated into the framework of the theory of relativity early on, in the form of **General Relativity** that today generalizes Newton's laws of gravitation. However, it remains until today one of the great unsolved problems in physics to unify General Relativity with quantum mechanics. Thus, general relativity is not a part of the SM of elementary particles. For the large majority of questions in particle physics, this is not a problem because gravity is so many orders of magnitude weaker than the other three forces of Nature. For questions involving quantum length scales and bodies with large masses – such as black holes or the very early universe – a quantum theory of general relativity seems indispensable.

This course, however, is aimed at phenomena in nuclear physics (and particle physics) where gravity is negligible. There is one aspect of general relativity, however, that will be used in the present context: The "language" of co- and contravariant four-tensors that will be developed (as far as required) in the first half of the course. This formalism is frowned upon by many students (and some teachers as well!) for the reason that it adds a level of complexity to an already difficult part of physics. However, modern fundamental physicists use this language everywhere (!), and so it is obligatory for us to learn it. The price of learning comes with immediate advantages, too, since it massively simplifies the understanding of Lorentz covariance.

A few examples from different fields of physics shall illustrate the importance of special relativity. They highlight the following relativistic phenomena and central aspects of this course:

- Mass $\leftrightarrow$ energy conversion (t production)
- Existence of antimatter (t production, PET treatment, Fermion EDM)
- Relativistic effects in bound matter (lead battery, Fermion EDM)
- Nuclear decays / radioactivity (PET treatment)
- Length contraction (Fermion EDM)

0.1.1 Special relativity in high-energy physics: Production of the *top*(t) quark

antimatter; mass-energy conversion

In 1995 the production of the so far heaviest elementary particle, the top (t) quark, succeeded at the Tevatron collider at Fermilab just outside Chicago. It was produced as and decays as shown in the image:

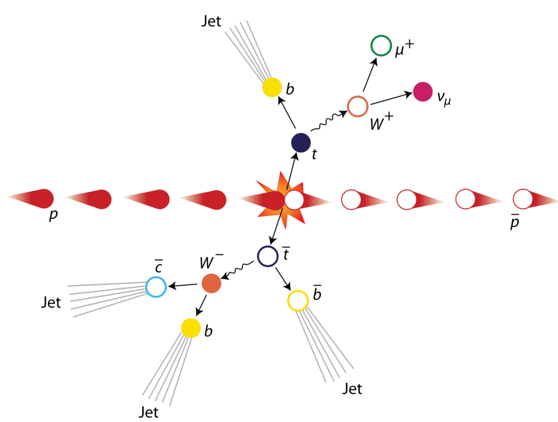


Figure 1:

A high-energy collision of a proton (p) and an antiproton ($\overline{p}$) produces a $t\overline{t}$ pair *via* the strong interaction. These have a lifetime of $\approx 10^{-25}$ [s] and rapidly decay into a bottom (b) quark and a W^+ vector boson (the $\bar{t}$ decays into an antibottom ($\bar{b}$) and a W^- vector boson). These in turn lead to further decays, the debris of which is detected at the facilities. Note that the particle electric charges are as follows (in elementary charges): $C(t) = +2/3$, $C(b) = -1/3$, $C(W^+) = +1$.

What is remarkable is that the rest mass of the t is $m(t) \approx 172 \left[\frac{\text{GeV}}{\text{c}^2}\right]$ whereas the sum of the rest masses of the proton and the antiproton is only $m(p) + m(\overline{p}) \approx 1.88 \left[\frac{\text{GeV}}{\text{c}^2}\right]$. The rest mass of a t roughly corresponds to the mass of a tungsten (W) atom!

So the mass of the incident particles constitute only about 0.5% of the mass of the produced particles. An explanation for this can only be given by one of the central theorems of special relativity which states that energy (in this case kinetic energy) can be converted into a different form of energy, in this case rest energy, representing rest mass $\times c^2$.

The production of the extremely heavy t quark is just one example of how special relativity acts in high-energy physics. Nowadays, New Physics searches, in particular for SuperSymmetric (SUSY) particles, probe into the energy range of ≈ 1000 [TeV] $= 1$ [PeV].

0.1.2 Nuclear Medicine and Special Relativity: PET Treatment

antimatter; radioactivity; mass-energy conversion

Positron Emission Tomography (PET) is used as a means to detect ("imaging") and treat cancerous cells in the human body. A "tracer" is injected which contains a radionuclide, e.g. ^{18}F*, in a fluorine atom bound in a biologically active molecule, ^{18}F*BAM. This radioactive nucleus in a nuclear excited state arrives at the targeted position and decays under β^+ (positron) emission:

$$p^* \longrightarrow n + e^+ + \nu_e \tag{1}$$

This is the fundamental process underlying the decay of ^{18}F*. Note that the rest mass of the proton is **smaller** than that of the neutron, but the proton is in an excited state, and it is this excitation energy that can be converted into rest energy! The positron does not survive for long because it will undergo **pair annihilation** with an electron of adjacent matter:

$$e^+ + e^- + X^+ \longrightarrow 2\gamma + X^+ \tag{2}$$

where the nucleus X^+ of the atom providing the bound electron has been included in the reaction. Pair annihilation[2] is a high-energy process with emitted photon energies $\varepsilon(\gamma)$ on the order of ≈ 500 [keV][3]. This energy is to be compared with typical atomic transition energies, on the order of $0.01 \ldots 10$ [eV], depending on the involved degrees of freedom (rovibrational, electronic). The high-energy radiation is then detected by detectors surrounding the patient.

So once again, we are confronted with the existence of antimatter. Its existence, in fact, of the positron, was one of the most spectacular *predictions* of theoretical physics, and it is a natural consequence of

[2]The amplitudes (transition probabilities) for such processes can be calculated with Feynman's formalism in the framework of QED.

[3]Such a photon energy converts via $E = h\nu$ and $\lambda = c/\nu$ into a wavelength of about 2.5 pm, constituting γ rays.

special relativity, becoming manifest in the **Dirac equation**, here written in **Lorentz covariant** form:

$$(-\imath\hbar\,\gamma^{\mu}\,\partial_{\mu} + m_0 c\,\mathbb{1}_4)\;\underline{\Psi}(x) = \underline{0} \tag{3}$$

We will walk through its derivation, its solutions, and its basic interpretation and consequences.

0.1.3 Atomic Matter and Special Relativity: Lead-Acid Battery

fundamental properties (spin); relativistic mass; magnetic interactions

As an example from the physics of ordinary atomic matter, consider the lead-acid battery. This work has been published in Phys. Rev. Lett. **106** (2011) *018301*.

The potential difference of a conventional lead-acid battery for cars is 12 [V]. The electrons produced at the negative pole give rise to an electrical current that launches the starter of the car.

The potential difference is obviously crucial for the functioning of the battery. The authors found that it is a function of the energy of electrons occupying the 6s state in lead, $U = U(\varepsilon(6s_{\rm Pb}))$. This energy, in turn, is different in a non-relativistic world, i.e., if special relativity is "turned off". The main reason for this energy difference is the difference between non-relativistic and **relativistic momentum** of the electron in the rest frame of the nucleus. In perturbation theory the total energy of a one-particle atomic state can be written as

$$\varepsilon = \varepsilon_{\rm n.r.} + \varepsilon^{(1)}_{\rm MV} + \varepsilon^{(1)}_{\rm Dar} \tag{4}$$

The relativistic correction "mass-velocity" and "Darwin" can be obtained from Pauli's approximation to Dirac theory, yielding

$$\begin{aligned} \varepsilon^{(1)}_{\rm MV} &= -\frac{\alpha^2 e^2}{a_0}\,\frac{Z^4_{\rm eff}}{8n^4}\left[\frac{4n}{\ell+\frac{1}{2}} - 3\right] \\ \varepsilon^{(1)}_{\rm Dar} &= \frac{\alpha^2 e^2}{a_0}\,\delta_{0\ell}\,\frac{Z^4_{\rm eff}}{2n^3} \end{aligned}$$

for single-particle states in an atom[4]. These relativistic corrections

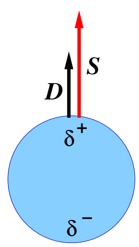
[4] $Z_{\rm eff}$ is the effective nuclear charge in the state, α is Sommerfeld's fine-structure constant, e is the elementary charge, a_0 is the Bohr radius, n is the state's principal quantum number, and ℓ is the orbital angular-momentum quantum number.

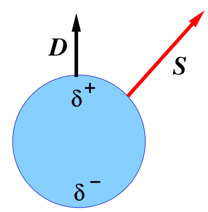
depend on the environment and are different in Pb and PbO_2. The authors were able to model the car battery without these relativistic effects, and the result was $U_{\text{non-rel}} \approx 10$ [V]. Such a tension is insufficient for launching the car's starter! So whenever you start your car, remember that this works because of special relativity.

0.1.4 Search for New Physics: Fermion Electric Dipole Moment

fundamental properties; antimatter; length contraction; magnetic interactions

It is well known that fermions have magnetic dipole moments since this is proportional to the spin of the fermion (its intrinsic angular momentum). First of all, particle spin is a relativistic phenomenon. We will see this in the context of the Dirac equation. But can fermions have an **electric** dipole moment (EDM)? If this were the case, the EDM vector could relate to the spin vector (operator) in two principal ways:

Why 

? and not 

?

We will not go into why the second scenario is impossible (it leads to an internal contradiction in Fermi-Dirac statistics). So what is the Hamiltonian for such a fundamental electric dipole in an external E field?

Classical electromagnetism:

$\varepsilon_{\mathrm{dip}} = -\mathbf{D}\cdot\mathbf{E}$

Fermion EDM vector operator $\hat{\mathbf{d}} \propto \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma} \end{pmatrix}$ and so[5]

$\hat{H}_{\mathrm{EDM}} = -d_f\,\gamma^0\,\boldsymbol{\Sigma}\cdot\mathbf{E}$

The proportionality constant d_f is the fermion EDM.

Dirac matrix γ^0 ensures that $\left\langle \hat{H} \right\rangle$ is a Lorentz scalar (we will see in a short while what that is.)

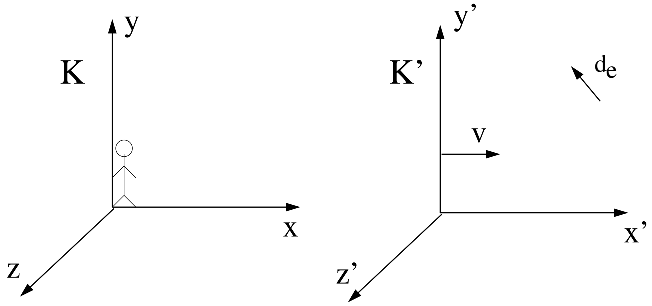
This energy $\left\langle \hat{H} \right\rangle$ is a $\mathcal{T}$-odd *pseudo*scalar.

The potential energy of a fermion EDM in an electric field ((incl.

[5]E. Salpeter, *Phys Rev* **112** (1958) *1642*

E_{ext}) for a state $\psi^{(0)}$ is thus the expectation value
$\varepsilon_{\text{EDM}} = \left\langle -d_e \gamma^0 \mathbf{\Sigma} \cdot \mathbf{E} \right\rangle_{\psi^{(0)}}$

Interpretation:[6]



Length contraction for collinear movement:
$\mathbf{d}_e(K) = \frac{\mathbf{d}_e(K')}{\gamma} = \mathbf{d}_e(K') \left(1 - \frac{\gamma}{1+\gamma} \frac{v^2}{c^2}\right)$

... and for general movement:
$\mathbf{d}_e(K) = \mathbf{d}_e(K') - \frac{\gamma}{1+\gamma} \frac{\mathbf{v}}{c} \left(\mathbf{d}_e(K') \cdot \frac{\mathbf{v}}{c}\right)$

The dipole energy in K then is
$\varepsilon_{\text{dip}} = -\mathbf{d}_e(K) \cdot \mathbf{E} = -\mathbf{d}_e(K') \cdot \left[\mathbf{E} - \frac{\gamma}{1+\gamma} \frac{\mathbf{v}}{c} \left(\frac{\mathbf{v}}{c} \cdot \mathbf{E}\right)\right]$
For small relative velocities we can approximate:
$\varepsilon_{\text{dip}} \approx -\mathbf{d}_e(K') \cdot \mathbf{E} + \frac{1}{2m_e^2 c^2} \mathbf{d}_e(K') \cdot \mathbf{p} \, (\mathbf{p} \cdot \mathbf{E})$

The QM expression can be approximated similarly, yielding

$\varepsilon_{\text{EDM}} \approx -d_e \left\{ \langle \boldsymbol{\sigma} \cdot \mathbf{E} \rangle_{\Psi^L} - \frac{1}{4m^2c^2} \left[\langle \hat{\mathbf{p}} \cdot \mathbf{E} \, \boldsymbol{\sigma} \cdot \hat{\mathbf{p}} \rangle_{\Psi^L} + \langle \mathbf{E} \cdot \hat{\mathbf{p}} \, \boldsymbol{\sigma} \cdot \hat{\mathbf{p}} \rangle_{\Psi^L} \right] \right\}$

which corresponds to the **classical dipole energy in the observer frame**.

0.1.5 Fundamental Physical Theory: Weak-Interaction Lagrangian

Nuclear physics; covariant formalism; quantum-field theory

[6] E.D. Commins, J.D. Jackson, D.P. DeMille, *Am J Phys* **75** (2007) *532*

0.2 Galilei Invariance

At the end of the 19th century classical mechanics was a theory, the equations of motion of which were invariant to so-called **Galilei transformations**. These include rotations in three-dimensional coordinate (real) space and "boosts". The ensemble of such transformations forms the **group of Galilei transformations**.

Let us review Newton's axioms:

1. There exists an **inertial frame**[7] **(or system) of reference** in which the forceless motion of a particle is described by a constant velocity,

$$\mathbf{v} = \mathbf{const.} \tag{5}$$

2. In inertial frames the motion of a particle under the influence of a force is described by the **equation of motion**

$$m\mathbf{a} = \sum_i \mathbf{F}_i = \dot{\mathbf{p}} \tag{6}$$

3. For every force with which a particle (1) acts on another particle (2) there is an equal and opposite reaction force with which particle (2) acts on particle (1):

$$\mathbf{F}_{1\to 2} = -\mathbf{F}_{2\to 1} \tag{7}$$

As an illustration of the mentioned invariance, we will explicitly transform the second axiom under boost:

A coordinate transformation between two inertial frames which move at a constant velocity relative to one another is called a **boost**. We will test the invariance of the fundamental law of dynamics for one of

[7]This course will be restricted to the treatment of inertial frames. Nevertheless, in Euclidean (flat) spacetime, it is possible to treat the non-inertial frame's acceleration as an acceleration seen in an inertial frame. It is thus possible to solve problems involving accelerated motion in the framework of special relativity.

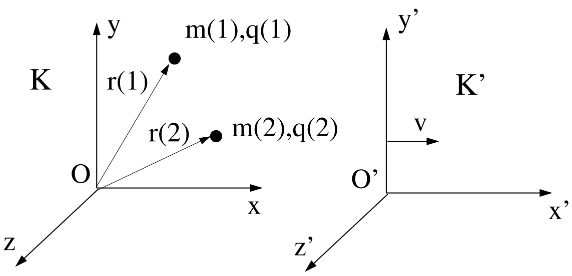


Figure 2:

Two interacting particles and two inertial frames related by a boost transformation. It is supposed that at $t = t' = 0$ we have $x = x' = 0$, *i.e.*, the origins of K and K' coincide.

the particles.

$$\begin{aligned} \mathbf{F}_{2\to 1} &= m(1)\mathbf{a}(1) \\ -\boldsymbol{\nabla}(1)\,\kappa\,\varphi_{21} &= m(1)\mathbf{a}(1) \\ -\kappa\sum_{j=1}^{3}\mathbf{e}_j\,\frac{\partial}{\partial x_j(1)}\,\varphi\left(||\mathbf{r}(1)-\mathbf{r}(2)||\right) &= m(1)\frac{d\mathbf{v}(1)}{dt} \end{aligned} \tag{8}$$

where we are considering a fundamental force that can be written as dependent on the negative gradient of some scalar potential (electrostatic, gravitational) that depends only on the distance between the particles, and κ is a constant (q_1 or m_1, respectively)[8].

The individual terms are now subjected to the Galilei transformation from inertial frame K to inertial frame K':

$$\begin{aligned} \mathbf{r}'(n) &= \mathbf{r}(n) - vt\mathbf{e}_x \\ t' &= t \end{aligned} \tag{12}$$

[8]Verification of the second line for the case of electromagnetic interaction: The electric potential at instant t and position of particle 1 due to the presence of particle 2 is

$$\varphi_{21} = \frac{1}{4\pi\varepsilon_0}\frac{q_2}{||\mathbf{r}(1)-\mathbf{r}(2)||}. \tag{9}$$

The gradient of this potential with respect to coordinates of particle 1 then is

$$\boldsymbol{\nabla}(1)\,\varphi_{21} = -\frac{q_2}{4\pi\varepsilon_0}\frac{\mathbf{r}(1)-\mathbf{r}(2)}{||\mathbf{r}(1)-\mathbf{r}(2)||^3}. \tag{10}$$

From this it follows for the electric field at position 1:

$$\mathbf{E}(\mathbf{r}(1)) = -\boldsymbol{\nabla}(1)\,\varphi_{21} = \frac{q_2}{4\pi\varepsilon_0}\frac{\mathbf{r}(1)-\mathbf{r}(2)}{||\mathbf{r}(1)-\mathbf{r}(2)||^3} \tag{11}$$

and so the force on particle 1 is correctly obtained as $\mathbf{F}_{2\to 1} = q_1\mathbf{E}(\mathbf{r}(1))$ and $\kappa = q_1$ in this case.

and we obtain in detail

- $\mathbf{a}'(1) = \frac{d\mathbf{v}'(1)}{dt'} = \frac{d^2\mathbf{r}'(1)}{dt'^2} = \frac{d^2}{dt^2}\left(\mathbf{r}(1) - vt\mathbf{e}_x\right) = \frac{d^2}{dt^2}\mathbf{r}(1) = \mathbf{a}(1)$
  Whereas velocity and momentum depend on the reference frame, acceleration does not.

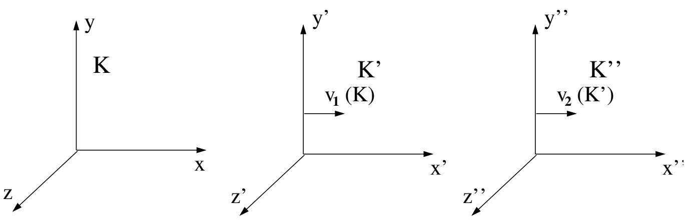
- The potential depends on distance between particles only, so we regard that distance:
  $||\mathbf{r}'(1) - \mathbf{r}'(2)|| = ||\mathbf{r}(1) - vt\mathbf{e}_x - \mathbf{r}(2) + vt\mathbf{e}_x|| = ||\mathbf{r}(1) - \mathbf{r}(2)||$. And so we can conclude that $\varphi' = \varphi$.

- For transforming the gradient we have to respect the functional relationship between the coordinates as due to the transformation, here $\mathbf{r} = \mathbf{r}' + vt\mathbf{e}_x$. Suppose the most general case of a (tensor) field f that is defined in space and time for the frame K, $f = f\big(x(x'), y(x'), z(x'), t(x')\big)$ where all the variables are generally functions of all of the coordinates of K' (x' is sufficient here). Then the total rate of change of f with respect to x' is expressed via the chain rule and we can write
  $\frac{\partial}{\partial x'(1)} = \frac{\partial}{\partial x(1)}\frac{\partial x(1)}{\partial x'(1)} + \frac{\partial}{\partial y(1)}\frac{\partial y(1)}{\partial x'(1)} + \frac{\partial}{\partial z(1)}\frac{\partial z(1)}{\partial x'(1)} + \frac{\partial}{\partial t}\frac{\partial t}{\partial x'(1)} = \frac{\partial}{\partial x(1)}$
  Since for the above Galilei transformation $\frac{\partial x(1)}{\partial x'(1)} = 1$, $\frac{\partial y(1)}{\partial x'(1)} = 0$, same for z, and $\frac{\partial t}{\partial x'(1)} = 0$ since $t = t'$ it follows that
  $\Rightarrow \boldsymbol{\nabla}_x = \boldsymbol{\nabla}_{x'}$.

which proves that $\mathbf{F}'_{2\to 1} = m(1)\mathbf{a}'(1)$ is **equivalent** (form invariant) to the untransformed law of motion. Note that particle mass is absolute in Newtonian physics, just like time is. Newtonian mechanics is said to be **Galilei (boost) invariant**.

Boosts are not the only conceivable kind of transformations between reference frames. In the case frame K' being rotated by an angle α about any particular axis relative to K, this angle plays the role of the constant velocity in the above demonstration. Newton's law is, therefore, also invariant under reference frame rotations.

From the Galilei transformation we can deduce the relative velocity between two inertial frames that move relatively to a third inertial frame in a known way. Fig. (1.1.3.1) shows the setup of the thought experiment.



Inertial frames K, K' and K" with axes aligned. Origins coincide at $t = t' = t'' = 0$. For instance, an observer may be standing on a platform at rest in K, a train (with v_1) carries a gunner who shoots off a bullet (with v_2).

Supposing that the relative velocities v_1 and v_2 are known, we wish to deduce the relative velocity v_3 between frames K and K". From the Galilei transformation in Eq. (12) using two successive Galilei boosts we have, written in matrix form which we will be useful for the treatment of the Einsteinian case,

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} 1 & -v_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} x - v_1 t \\ t \end{pmatrix} \tag{13}$$

$$\begin{pmatrix} x'' \\ t'' \end{pmatrix} = \begin{pmatrix} 1 & -v_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix} \tag{14}$$

Inserting Eq. (13) into Eq. (14) yields

$$\begin{aligned} \begin{pmatrix} x'' \\ t'' \end{pmatrix} &= \begin{pmatrix} 1 & -v_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -v_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \\ &= \begin{pmatrix} 1 & -v_1 - v_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}. \end{aligned} \tag{15}$$

Preserving the form of transformation means that the relative velocity between frames K and K" has to be $v_3 := v_1 + v_2$. This result is generalized to the **theorem of addition of velocities**:

$$\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2. \tag{16}$$

Therefore, assuming for simplicity that the involved velocities are collinear, a light beam's velocity ($v_2 = c$) emitted from an approaching train with velocity v_1 relative to a stationary observer[9] measured in the reference frame of this observer should give the result

$$v_3 = v_1 + c \tag{17}$$

But experiments[10] with sufficient accuracy carried out in the 20th century agree on the fact that the speed of light in all reference frames is $v_3 = c$, irrespective of the magnitude of v_2, i.e., the state of movement of the light source!

This finding has been elevated to become a **postulate** by A. Einstein in his seminal paper[11] from 1905. It will be invoked as the fourth and last postulate in the axiomatic construction of the special theory of relativity.

[9] Remaining in the earlier picture, imagine that the gunner is now using a laser gun.

[10] A. A. Michelson and E. W. Morley, *Am J Sci* **34** (1887) *333-345*
G. Joos, *Ann Phys* **7** (1930) *385*

[11] A. Einstein, *Ann Phys* **17** (1905) *891-921*