1 Differential Operators in Curvilinear Coordinates

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using LaTeX and git v1.7.3.4.

1.1 Generalities

We will here in particular be interested in transformations of certain classes of (differential) operators from linear to curvilinear coordinates.

1.1.1 Coordinates

Be $\{\alpha_j\}, j \in \{1 \ldots 3\}$ an arbitrary set of linear or curvilinear coordinates[1]. We express a position vector in a 3-dimensional vector space as

$$\vec{x} = \sum_{j=1}^{3} \alpha_j \vec{e}_j \tag{1}$$

and the total differential of the position vector accordingly as

$$d\vec{x} = \sum_{j=1}^{3} \frac{\partial \vec{x}}{\partial \alpha_j} d\alpha_j \tag{2}$$

1.1.2 Scalar Fields

Be f a scalar and differentiable field, allowing us to write the total differential in terms of a set of coordinates

$$df = \sum_{j=1}^{3} \frac{\partial f}{\partial \alpha_j} d\alpha_j \tag{3}$$

1.2 Cylindrical Coordinates

We define a local orthonormal basis of cylindrical coordinates as in Fig. 1.2 which are related to cartesian coordinates as follows: Accordingly we obtain from elementary geometrical considerations:

$$\begin{aligned} x &= \rho \cos \varphi && (4)\\ y &= \rho \sin \varphi && (5)\\ z &= z && (6) \end{aligned}$$

[1] coordinate patches on differentiable manifolds

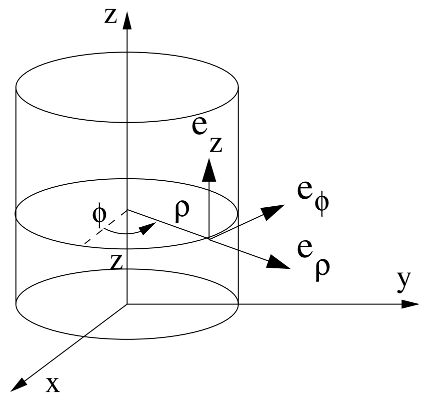


Figure 1: Cylindrical coordinates

1.2.1 Elementary transformations and position vectors

A position vector can therefore be written in terms of cylindrical coordinates as

$$\vec{x} = (\rho\cos\varphi)\,\vec{e}_x + (\rho\sin\varphi)\,\vec{e}_y + (z)\,\vec{e}_z \tag{7}$$

and we ensuingly obtain

$$\begin{aligned}
\frac{\partial\vec{x}}{\partial\rho} &= (\cos\varphi)\,\vec{e}_x + (\sin\varphi)\,\vec{e}_y \\
\frac{\partial\vec{x}}{\partial\varphi} &= -(\rho\sin\varphi)\,\vec{e}_x + (\rho\cos\varphi)\,\vec{e}_y \\
\frac{\partial\vec{x}}{\partial z} &= \vec{e}_z
\end{aligned} \tag{8}$$

and therefore the corresponding norms (scaling factors)

$$\begin{aligned}
\left|\left|\frac{\partial\vec{x}}{\partial\rho}\right|\right| &= 1 := h_1 \\
\left|\left|\frac{\partial\vec{x}}{\partial\varphi}\right|\right| &= \rho := h_2 \\
\left|\left|\frac{\partial\vec{x}}{\partial z}\right|\right| &= 1 := h_3
\end{aligned} \tag{9}$$

This allows us to write the orthonormal local tripod in terms of the new cylindrical coordinates as

$$\begin{aligned}
\vec{e}_\rho &= \frac{\frac{\partial\vec{x}}{\partial\rho}}{\left|\left|\frac{\partial\vec{x}}{\partial\rho}\right|\right|} = (\cos\varphi)\,\vec{e}_x + (\sin\varphi)\,\vec{e}_y \\
\vec{e}_\varphi &= \frac{\frac{\partial\vec{x}}{\partial\varphi}}{\left|\left|\frac{\partial\vec{x}}{\partial\varphi}\right|\right|} = -(\sin\varphi)\,\vec{e}_x + (\cos\varphi)\,\vec{e}_y \\
\vec{e}_z &= \frac{\frac{\partial\vec{x}}{\partial z}}{\left|\left|\frac{\partial\vec{x}}{\partial z}\right|\right|} = \vec{e}_z
\end{aligned} \tag{10}$$

Consequently, the original tripod can be expressed as

$$\begin{aligned} \vec{e}_x &= (\cos\varphi)\,\vec{e}_\rho - (\sin\varphi)\,\vec{e}_\varphi \\ \vec{e}_y &= (\sin\varphi)\,\vec{e}_\rho + (\cos\varphi)\,\vec{e}_\varphi \\ \vec{e}_z &= \vec{e}_z \end{aligned} \tag{11}$$

Finally we rewrite the position vector from Eq. (7) as

$$\vec{x} = \rho\vec{e}_\rho + z\vec{e}_z \tag{12}$$

and the total differential from Eqs. (2), (8) and (11) as

$$\begin{aligned} d\vec{x} &= \frac{\partial\vec{x}}{\partial\rho}d\rho + \frac{\partial\vec{x}}{\partial\varphi}d\varphi + \frac{\partial\vec{x}}{\partial z}dz \\ &= d\rho\;\vec{e}_\rho + \rho d\varphi\;\vec{e}_\varphi + dz\;\vec{e}_z \end{aligned} \tag{13}$$

1.2.2 Line Elements

Elements for multidimensional integration can be easily obtained from the determined scaling factors, Eq. (9). We obtain for line elements

$$\begin{aligned} dl_\rho &= h_1 d\rho = d\rho \\ dl_\varphi &= h_2 d\varphi = \rho\, d\varphi \\ dl_z &= h_3 dz = dz, \end{aligned} \tag{14}$$

surface elements for integration over a slice (orthogonal or longitudinal) or the curved surface of the cylinder, respectively,

$$d\mathcal{S}_{\text{o-slice}} = dl_\rho\, dl_\varphi = h_1 h_2\, d\rho d\varphi = \rho\, d\rho d\varphi$$

$$d\mathcal{S}_{\text{l-slice}} = dl_\rho\, dl_z = h_1 h_3\, d\rho dz = d\rho dz \tag{15}$$

$$d\mathcal{S}_{\text{surface}} = dl_\varphi dl_z = h_2 h_3\, d\varphi dz = \rho\, d\varphi dz \tag{16}$$

and finally for the volume element

$$d\mathcal{V} = dl_\rho\, dl_\varphi dl_z = h_1 h_2 h_3\, d\rho d\varphi dz = \rho\, d\rho d\varphi dz. \tag{17}$$

1.2.3 The Gradient Operator

In terms of cylindrical coordinates the gradient operator $\vec{grad} = \vec{\nabla}$ can be written as

$$\vec{\nabla} f = \left(\vec{\nabla} f\right)_\rho \vec{e}_\rho + \left(\vec{\nabla} f\right)_\varphi \vec{e}_\varphi + \left(\vec{\nabla} f\right)_z \vec{e}_z \tag{18}$$

Since $\vec{\nabla}$ can be related to the total differential of a scalar field according to $df = \vec{\nabla} f\; d\vec{x}$ (which in a slightly improper but intuitive form can be written as $\frac{df}{d\vec{x}} = \vec{\nabla} f$). Using the result from Eq. (13) we obtain

$$\vec{\nabla} f\; d\vec{x} = \left(\vec{\nabla} f\right)_\rho d\rho + \rho\left(\vec{\nabla} f\right)_\varphi d\varphi + \left(\vec{\nabla} f\right)_z dz \tag{19}$$

We now employ the general form of the total differential of the scalar field Eq. (3)

$$df = \frac{\partial f}{\partial \rho} d\rho + \frac{\partial f}{\partial \varphi} d\varphi + \frac{\partial f}{\partial z} dz \tag{20}$$

and compare coefficients between Eqs. (19) and (20) to obtain

$$\begin{aligned} \frac{\partial f}{\partial \rho} &= \left(\vec{\nabla} f\right)_\rho \\ \frac{\partial f}{\partial \varphi} &= \rho \left(\vec{\nabla} f\right)_\varphi \\ \frac{\partial f}{\partial z} &= \left(\vec{\nabla} f\right)_z \end{aligned} \tag{21}$$

This yields the gradient operator in cylindrical coordinates acting on a scalar field

$$\boxed{\vec{\nabla} f = \vec{e}_\rho \frac{\partial f}{\partial \rho} + \vec{e}_\varphi \frac{1}{\rho} \frac{\partial f}{\partial \varphi} + \vec{e}_z \frac{\partial f}{\partial z}} \tag{22}$$

We realize that the gradient operator in curvilinear coordinates can in general be written as

$$\vec{\nabla} f = \sum_{j=1}^{3} \vec{e}_j \frac{1}{h_j} \frac{\partial f}{\partial \alpha_j} \tag{23}$$

where $h_j = \left|\frac{\partial \vec{x}}{\partial \alpha_j}\right|$ are scaling factors in the respective coordinate system (for example in cylindrical coordinates they are given in Eq. (9)). This is also readily verified in cartesian coordinates.

1.2.4 The Divergence Operator

Once the gradient operator in curvilinear coordinates is known, the divergence operator acting on a vector field can be deduced in the following manner:

We introduce an auxiliary scalar field $g = g(x_1, x_2, x_3) = g(\rho, \varphi, x_3)$ and calculate the integral over the product of the test vector field $\vec{F}$ and $gr\vec{a}dg$

$$\int_V \vec{F}(x_1, x_2, x_3) \cdot gr\vec{a}dg(x_1, x_2, x_3)\, dV = \int_V \vec{F}(\rho, \varphi, x_3) \cdot gr\vec{a}dg(\rho, \varphi, x_3)\, dV \tag{24}$$

both in cartesian and cylindrical coordinates, remembering that the result must of course be identical. For the left-hand side of Eq. (24) we obtain straightforwardly

$$\begin{aligned} \int_V \vec{F} \cdot gr\vec{a}dg\, dV &= \int\int\int F_1 \frac{\partial g}{\partial x_1}\, dx_1 dx_2 dx_3 + \int\int\int F_2 \frac{\partial g}{\partial x_2}\, dx_1 dx_2 dx_3 + \int\int\int F_3 \frac{\partial g}{\partial x_3}\, dx_1 dx_2 dx_3 \\ &= \int\int F_1 g\, dx_2 dx_3 - \int\int\int \frac{\partial F_1}{\partial x_1} g\, dx_1 dx_2 dx_3 \\ &\quad + \int\int F_2 g\, dx_1 dx_3 - \int\int\int \frac{\partial F_2}{\partial x_2} g\, dx_1 dx_2 dx_3 \\ &\quad + \int\int F_3 g\, dx_1 dx_2 - \int\int\int \frac{\partial F_3}{\partial x_3} g\, dx_1 dx_2 dx_3 \qquad (25) \\ &= -\int\int\int (\mathrm{div}\vec{F}) g\, dx_1 dx_2 dx_3 \qquad (26) \end{aligned}$$

choosing g such that the terms integrated by parts vanish[2]. The right-hand-side of Eq. (24) becomes

$$
\begin{aligned}
\int_V \vec{F}\cdot \vec{grad} g\, dV &= \int\int\int \vec{F}(\rho,\varphi,z)\left(\vec{e}_\rho \frac{\partial g}{\partial \rho} + \frac{1}{\rho}\vec{e}_\varphi \frac{\partial g}{\partial \varphi} + \vec{e}_z \frac{\partial g}{\partial z}\right)\rho d\rho d\varphi dz \\
&= \int\int\int \rho F_\rho \frac{\partial g}{\partial \rho}\, d\rho d\varphi dz + \int\int\int \frac{1}{\rho} F_\varphi \frac{\partial g}{\partial \varphi}\, \rho d\rho d\varphi dz + \int\int\int F_z \frac{\partial g}{\partial z}\, \rho d\rho d\varphi dz \\
&= \int\int \rho F_\rho g\, d\varphi dz - \int\int\int \left(F_\rho g + \rho \frac{\partial F_\rho}{\partial \rho} g\right) d\rho d\varphi dz \\
&\quad + \int\int F_\varphi g\, d\rho dz - \int\int\int \frac{\partial F_\varphi}{\partial \varphi} g\, d\rho d\varphi dz \\
&\quad + \int\int \rho F_z g\, d\rho d\varphi - \int\int\int \rho \frac{\partial F_z}{\partial z} g\, d\rho d\varphi dz \\
&= -\int\int\int \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho F_\rho\right) g\, \rho d\rho d\varphi dz \\
&\quad -\int\int\int \frac{1}{\rho}\frac{\partial F_\varphi}{\partial \varphi} g\, \rho d\rho d\varphi dz \\
&\quad -\int\int\int \frac{\partial F_z}{\partial z} g\, \rho d\rho d\varphi dz \\
&= -\int\int\int \left(\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho F_\rho\right) + \frac{1}{\rho}\frac{\partial F_\varphi}{\partial \varphi} + \frac{\partial F_z}{\partial z}\right) g\, \rho d\rho d\varphi dz \qquad (27)
\end{aligned}
$$

with vanishing terms integrated by parts. From Eqs. (26) and (27) the identity of the integrands yields the divergence of the vector field $\vec{F}$ in cylindrical coordinates

$$\boxed{\vec{\nabla}\cdot\vec{F} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho F_\rho\right) + \frac{1}{\rho}\frac{\partial F_\varphi}{\partial \varphi} + \frac{\partial F_z}{\partial z}} \qquad (28)$$

1.2.5 The Curl Operator

Here we want to deduce a general expression for the curl operator in any locally orthonormal curvilinear coordinate system, and then extract special cases such as cylindrical coordinates. We express the curl of a vector field $\vec{F}$ as

$$
\begin{aligned}
\vec{rot}\,\vec{F} &= \vec{\nabla}\times\vec{F} = \vec{\nabla}\times\left(\sum_{i=1}^{3} F_j \vec{e}_j\right) \\
&= \vec{\nabla}\times(F_1\vec{e}_1) + \vec{\nabla}\times(F_2\vec{e}_2) + \vec{\nabla}\times(F_3\vec{e}_3) \qquad (29)
\end{aligned}
$$

due to the linearity of the vector product. Replacing f by α_k in Eq. (23) we infer

$$\vec{\nabla}\alpha_k = \sum_{j=1}^{3} \vec{e}_j \frac{1}{h_j}\frac{\partial \alpha_k}{\partial \alpha_j} = \sum_{j=1}^{3} \vec{e}_j \frac{1}{h_j}\delta_{jk} = \vec{e}_k \frac{1}{h_k} \qquad (30)$$

[2] TODO: Some more detailed mathematical comments are required on this point.

and so the first term of Eq. (29) can be written as

$$\vec{\nabla} \times (F_1 \vec{e}_1) = \vec{\nabla} \times \left(F_1 h_1 \vec{\nabla} \alpha_1 \right). \tag{31}$$

We now employ the following identity[3]

$$\vec{rot}\left(\varphi \vec{F}\right) = \left(\vec{grad}\varphi\right) \times \vec{F} + \varphi \left(\vec{rot}\vec{F}\right) \tag{32}$$

which allows us to reformulate Eq. (31)

$$\vec{\nabla} \times (F_1 \vec{e}_1) = \left[\vec{\nabla}(F_1 h_1)\right] \times \left(\vec{\nabla}\alpha_1\right) + F_1 h_1 \left(\vec{\nabla} \times \vec{\nabla}\alpha_1\right). \tag{33}$$

Since the curl of a gradient vanishes, $\vec{rot}\,\vec{grad} f = 0$ for any scalar field f, Eq. (33) reduces to

$$\vec{\nabla} \times (F_1 \vec{e}_1) = \left[\vec{\nabla}(F_1 h_1)\right] \times \left(\vec{\nabla}\alpha_1\right). \tag{34}$$

We now substitute Eq. (30) into Eq. (34) and obtain

$$\vec{\nabla} \times (F_1 \vec{e}_1) = \left[\vec{\nabla}(F_1 h_1)\right] \times \frac{\vec{e}_1}{h_1}. \tag{35}$$

We now use the general expression of the gradient given in Eq. (23) rewritten for the scalar field $F_1 h_1$ to reformulate Eq. (35) to

$$\begin{aligned} \vec{\nabla} \times (F_1 \vec{e}_1) &= \left[\sum_{j=1}^{3} \vec{e}_j \frac{1}{h_j} \frac{\partial (F_1 h_1)}{\partial \alpha_j}\right] \times \frac{\vec{e}_1}{h_1} \\ &= \vec{e}_2 \frac{1}{h_3} \frac{\partial (F_1 h_1)}{\partial \alpha_3} \frac{1}{h_1} - \vec{e}_3 \frac{1}{h_2} \frac{\partial (F_1 h_1)}{\partial \alpha_2} \frac{1}{h_1} \end{aligned} \tag{36}$$

In an analogous fashion the two remaining terms of Eq. (29) are obtained as

$$\vec{\nabla} \times (F_2 \vec{e}_2) = -\vec{e}_1 \frac{1}{h_3} \frac{\partial (F_2 h_2)}{\partial \alpha_3} \frac{1}{h_2} + \vec{e}_3 \frac{1}{h_1} \frac{\partial (F_2 h_2)}{\partial \alpha_1} \frac{1}{h_2} \tag{37}$$

$$\vec{\nabla} \times (F_3 \vec{e}_3) = -\vec{e}_2 \frac{1}{h_1} \frac{\partial (F_3 h_3)}{\partial \alpha_1} \frac{1}{h_3} + \vec{e}_1 \frac{1}{h_2} \frac{\partial (F_3 h_3)}{\partial \alpha_2} \frac{1}{h_3} \tag{38}$$

Adding the three terms yields the general expression of the curl operator

$$\begin{aligned} \vec{rot}\vec{F} &= \vec{\nabla} \times \vec{F} \\ &= \frac{\vec{e}_1}{h_2 h_3} \left(\frac{\partial (F_3 h_3)}{\partial \alpha_2} - \frac{\partial (F_2 h_2)}{\partial \alpha_3} \right) \\ &\quad + \frac{\vec{e}_2}{h_1 h_3} \left(\frac{\partial (F_1 h_1)}{\partial \alpha_3} - \frac{\partial (F_3 h_3)}{\partial \alpha_1} \right) \\ &\quad + \frac{\vec{e}_3}{h_1 h_2} \left(\frac{\partial (F_2 h_2)}{\partial \alpha_1} - \frac{\partial (F_1 h_1)}{\partial \alpha_2} \right) \\ &= \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \vec{e}_1 & h_2 \vec{e}_2 & h_3 \vec{e}_3 \\ \frac{\partial}{\partial \alpha_1} & \frac{\partial}{\partial \alpha_2} & \frac{\partial}{\partial \alpha_3} \\ F_1 h_1 & F_2 h_2 & F_3 h_3 \end{vmatrix} \end{aligned} \tag{39}$$

[3] This equation can easily be proven to be correct in cartesian coordinates. Since we do not want general expressions to depend on coordinate frames, it has to be valid in general.

where we have written the curl conveniently using a determinant. Note that the term $h_1 h_2 h_3$ in the prefactor is just the determinant of the Jacobian matrix for the coordinate transformation.

Eq. (39) is a powerful and general expression from which the explicit form of the curl operator can be deduced with ease for different coordinate systems. In the present case of cylindrical coordinates, the scaling factors are $h_1 = 1$, $h_2 = \rho$, and $h_3 = 1$ and so the curl of a vector field $\vec{F}$ becomes

$$\boxed{\vec{\nabla} \times \vec{F} = \vec{e}_\rho \left(\frac{1}{\rho} \frac{\partial (F_z)}{\partial \varphi} - \frac{\partial (F_\varphi)}{\partial z} \right) + \vec{e}_\varphi \left(\frac{\partial (F_\rho)}{\partial z} - \frac{\partial (F_z)}{\partial \rho} \right) + \vec{e}_z \frac{1}{\rho} \left(\frac{\partial (\rho F_\varphi)}{\partial \rho} - \frac{\partial (F_\rho)}{\partial \varphi} \right)} \tag{40}$$

in cylindrical coordinates.

1.2.6 The Laplacian Operator

By definition, the Laplacian operator of a scalar field f is given as

$$\Delta f := \mathrm{div}(\vec{grad} f) \tag{41}$$

Using the expression in Eq. (28) for the divergence of a vector field $\vec{grad} f$ we obtain

$$\Delta f := \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho (\vec{grad} f)_\rho \right) + \frac{1}{\rho} \frac{\partial}{\partial \varphi} (\vec{grad} f)_\varphi + \frac{\partial}{\partial z} (\vec{grad} f)_z . \tag{42}$$

From Eq. (22) the components of the gradient of f are

$$\begin{aligned} (\vec{grad} f)_\rho &= \frac{\partial f}{\partial \rho} \\ (\vec{grad} f)_\varphi &= \frac{1}{\rho} \frac{\partial f}{\partial \varphi} \\ (\vec{grad} f)_z &= \frac{\partial f}{\partial z} \end{aligned} \tag{43}$$

and therefore the Laplacian operator becomes

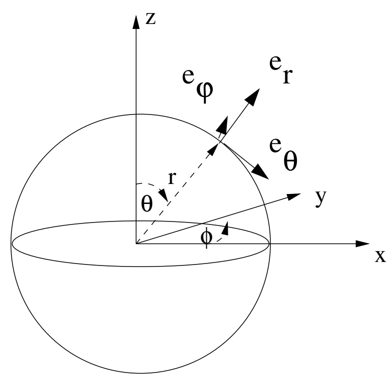
$$\Delta f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho} \frac{\partial}{\partial \varphi} \frac{1}{\rho} \frac{\partial f}{\partial \varphi} + \frac{\partial}{\partial z} \frac{\partial f}{\partial z} \tag{44}$$

or

$$\boxed{\Delta f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}} \tag{45}$$

1.3 3-Dimensional Spherical Polar Coordinates

Central-field problems are typically formulated and solved in spherical polar coordinates. We define from elementary geometrical considerations



$$
\begin{aligned}
x &= r\sin\vartheta\cos\varphi \\
y &= r\sin\vartheta\sin\varphi \\
z &= r\cos\vartheta
\end{aligned} \tag{46}
$$

which allows us to write a position vector as:

$$
\vec{x} = (r\sin\vartheta\cos\varphi)\,\vec{e}_x + (r\sin\vartheta\sin\varphi)\,\vec{e}_y + (r\cos\vartheta)\,\vec{e}_z \tag{47}
$$

1.3.1 Elementary transformations and position vectors

It follows

$$
\begin{aligned}
\frac{\partial\vec{x}}{\partial r} &= (\sin\vartheta\cos\varphi)\,\vec{e}_x + (\sin\vartheta\sin\varphi)\,\vec{e}_y + (\cos\vartheta)\,\vec{e}_z \\
\frac{\partial\vec{x}}{\partial\vartheta} &= (r\cos\vartheta\cos\varphi)\,\vec{e}_x + (r\cos\vartheta\sin\varphi)\,\vec{e}_y - (r\sin\vartheta)\,\vec{e}_z \\
\frac{\partial\vec{x}}{\partial\varphi} &= -(r\sin\vartheta\sin\varphi)\,\vec{e}_x + (r\sin\vartheta\cos\varphi)\,\vec{e}_y
\end{aligned} \tag{48}
$$

and therefore the scaling factors become

$$
\begin{aligned}
\left\|\frac{\partial\vec{x}}{\partial r}\right\| &= 1 := h_1 \\
\left\|\frac{\partial\vec{x}}{\partial\vartheta}\right\| &= r := h_2 \\
\left\|\frac{\partial\vec{x}}{\partial\varphi}\right\| &= r\sin\vartheta := h_3.
\end{aligned} \tag{49}
$$

The orthonormal local tripods can thus be deduced as:

$$
\begin{aligned}
\vec{e}_r &= \frac{\frac{\partial\vec{x}}{\partial r}}{\left\|\frac{\partial\vec{x}}{\partial r}\right\|} = \sin\vartheta\cos\varphi\vec{e}_x + \sin\vartheta\sin\varphi\vec{e}_y + \cos\vartheta\vec{e}_z \\
\vec{e}_\vartheta &= \frac{\frac{\partial\vec{x}}{\partial\vartheta}}{\left\|\frac{\partial\vec{x}}{\partial\vartheta}\right\|} = \cos\vartheta\cos\varphi\vec{e}_x + \cos\vartheta\sin\varphi\vec{e}_y - \sin\vartheta\vec{e}_z \\
\vec{e}_\varphi &= \frac{\frac{\partial\vec{x}}{\partial\varphi}}{\left\|\frac{\partial\vec{x}}{\partial\varphi}\right\|} = -\sin\varphi\vec{e}_x + \cos\varphi\vec{e}_y
\end{aligned} \tag{50}
$$

The original tripod is then expressed as:

$$\begin{aligned}
\vec{e}_x &= \sin\vartheta\cos\varphi\vec{e}_r + \cos\varphi\cos\vartheta\vec{e}_\vartheta - \sin\varphi\vec{e}_\varphi \\
\vec{e}_y &= \sin\vartheta\sin\varphi\vec{e}_r + \sin\varphi\cos\vartheta\vec{e}_\vartheta + \cos\varphi\vec{e}_\varphi \\
\vec{e}_z &= \cos\vartheta\vec{e}_r - \sin\vartheta\vec{e}_\vartheta \qquad (51)
\end{aligned}$$

We obtain for a position vector in spherical polar coordinates

$$\vec{x} = r\vec{e}_r \qquad (52)$$

by using Eqs. (47) and (51). In the following we deduce, in analogy with subsection 1.2.3 the differential operators in spherical polar coordinates.

1.3.2 Line Elements

Elements for multidimensional integration can be easily obtained from the determined scaling factors, Eq. (49). We obtain for line elements

$$\begin{aligned}
dl_r &= h_1 dr = dr \\
dl_\vartheta &= h_2 d\vartheta = r\,d\vartheta \\
dl_\varphi &= h_3 d\varphi = r\,\sin\vartheta\,d\varphi. \qquad (53)
\end{aligned}$$

The interesting surface element is the one over the curved sphere, for which we obtain

$$d\mathcal{S}_{\text{surface}} = dl_\vartheta dl_\varphi = h_2 h_3\,d\vartheta d\varphi = r^2\,\sin\vartheta\,d\vartheta d\varphi. \qquad (54)$$

The other possibilities are simply slices of the sphere which can be treated by using the calculus from cylindrical coordinates. The volume element finally reads

$$d\mathcal{V} = dl_r\,dl_\vartheta dl_\varphi = h_1 h_2 h_3\,dr d\vartheta d\varphi = r^2\,\sin\vartheta\,dr d\vartheta d\varphi. \qquad (55)$$

1.3.3 The Gradient Operator

With the help of the general expression for the gradient operator in curvilinear coordinates, Eq. (23), and the obtained scaling factors in Eqs. (49) the gradient operator can be formulated without additional calculation:

$$\boxed{\vec{\nabla} f = \vec{e}_r \frac{\partial f}{\partial r} + \vec{e}_\vartheta \frac{1}{r}\frac{\partial f}{\partial \vartheta} + \vec{e}_\varphi \frac{1}{r\sin\vartheta}\frac{\partial f}{\partial \varphi}} \qquad (56)$$

1.3.4 The Divergence Operator

As expounded in subsection 1.2.4 we start from an expression

$$\begin{aligned}
-\int\int\int \mathrm{div}\vec{F} g\ dx_1 dx_2 dx_3 &= \int\int\int \vec{F}(r,\vartheta,\varphi)\cdot \vec{grad} g\ dV \qquad (57) \\
&= \int\int\int \vec{F}(r,\vartheta,\varphi)\ \left(\vec{e}_r \frac{\partial g}{\partial r} + \frac{1}{r}\vec{e}_\vartheta \frac{\partial g}{\partial \vartheta} + \frac{1}{r\sin\vartheta}\vec{e}_\varphi \frac{\partial g}{\partial \varphi}\right)\ r^2 \sin\vartheta dr d\vartheta d\varphi
\end{aligned}$$

where the expression in Eq. (56) has been used for writing the gradient. Integrating by parts and comparing with the left-hand side of Eq. (57) we deduce the final expression for the divergence in spherical polar coordinates:

$$\boxed{\vec{\nabla}\cdot\vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta\, F_\vartheta) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\varphi}F_\varphi} \tag{58}$$

1.3.5 The Curl Operator

We use the second form in Eq. (39) as a starting point for deriving the curl operator. In spherical polar coordinates the scaling factors are determined as $h_1 = 1$, $h_2 = r$, and $h_3 = r\sin\vartheta$, and we immediately find

$$\boxed{\vec{\nabla}\times\vec{F} = \vec{e}_r\frac{1}{r\sin\vartheta}\left(\frac{\partial}{\partial\vartheta}(\sin\vartheta F_\varphi) - \frac{\partial F_\vartheta}{\partial\varphi}\right) + \vec{e}_\vartheta\frac{1}{r}\left(\frac{1}{\sin\vartheta}\frac{\partial F_r}{\partial\varphi} - \frac{\partial}{\partial r}(r\, F_\varphi)\right) + \vec{e}_\varphi\frac{1}{r}\left(\frac{\partial}{\partial r}(r F_\vartheta) - \frac{\partial F_r}{\partial\vartheta}\right)} \tag{59}$$

1.3.6 The Laplacian Operator

We again determine the Laplacian operator as $\Delta f := \mathrm{div}(\vec{grad} f)$. Using the expression for the divergence in Eq. (58) and replacing the vector field by the gradient of a scalar field f, given in Eq. (56) the Laplacian operator is derived as

$$\boxed{\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial f}{\partial\vartheta}\right) + \frac{1}{r^2(\sin\vartheta)^2}\frac{\partial^2 f}{\partial\varphi^2}} \tag{60}$$

Note that the radial part is often written differently, but the form given here is the one most commonly used in the literature.

1.4 4-Dimensional Spherical Polar Coordinates

The definition of spherical coordinates in 4-dimensional real space (R^4) can be inferred by analogy from the 2-dim. and 3-dim. cases:

$$\begin{aligned} x_1 &= r\cos\varphi_1 \\ x_2 &= r\sin\varphi_1\cos\varphi_2 \\ x_3 &= r\sin\varphi_1\sin\varphi_2\cos\varphi_3 \\ x_4 &= r\sin\varphi_1\sin\varphi_2\sin\varphi_3 \end{aligned} \tag{61}$$

which allows us to write a position vector as:

1.4.1 Elementary transformations and position vectors

It follows and therefore the scaling factors become The orthonormal local tripods can thus be deduced as: The original tripod is then expressed as: We obtain for a position vector in spherical polar coordinates

1.4.2 Line Elements

Elements for multidimensional integration can be easily obtained from the determined scaling factors, Eq. (49). We obtain for line elements The interesting surface element is the one over the curved sphere, for which we obtain The other possibilities are simply slices of the sphere which can be treated by using the calculus from cylindrical coordinates. The volume element finally reads

1.4.3 The Gradient Operator

1.4.4 The Divergence Operator

where the expression in Eq. (56) has been used for writing the gradient. Integrating by parts and comparing with the left-hand side of Eq. (57) we deduce the final expression for the divergence in spherical polar coordinates:

1.4.5 The Curl Operator

1.4.6 The Laplacian Operator

We again determine the Laplacian operator as $\Delta f := \mathrm{div}(\vec{grad} f)$. Using the expression for the divergence in Eq. (58) and replacing the vector field by the gradient of a scalar field f, given in Eq. (56) the Laplacian operator is derived as

1.5 Elliptical Coordinates

We define a local orthonormal basis of elliptical coordinates as in Fig. **??** which are related to cartesian coordinates as follows:

$$x = a \cosh\mu \cos\nu \tag{62}$$

$$y = a \sinh\mu \sin\nu \tag{63}$$

1.5.1 Elementary transformations and position vectors

A position vector can therefore be written in terms of elliptical coordinates as

$$\vec{x} = (a \cosh\mu \cos\nu)\,\vec{e}_x + (a \sinh\mu \sin\nu)\,\vec{e}_y \tag{64}$$

and we ensuingly obtain

$$\begin{aligned} \frac{\partial \vec{x}}{\partial \mu} &= (a \sinh\mu \cos\nu)\,\vec{e}_x + (a \cosh\mu \sin\nu)\,\vec{e}_y \\ \frac{\partial \vec{x}}{\partial \nu} &= (-\cosh\mu \sin\nu)\,\vec{e}_x + (a \sinh\mu \cos\nu)\,\vec{e}_y \end{aligned}$$

Using the relation

$$\sinh^2\mu\cos^2\nu+\cosh^2\mu\sin^2\nu=\sinh^2\mu+\sin^2\nu \tag{65}$$

which can be obtained with the help of

$$\begin{aligned}\cosh^2\mu-\sinh^2\mu &= 1\\ \sin^2\nu+\cos^2\nu &= 1\end{aligned}$$

we get

$$\begin{aligned}\left|\left|\frac{\partial\vec{x}}{\partial\mu}\right|\right| &= a\left(\sinh^2\mu+\sin^2\nu\right)^{\frac{1}{2}} := h_1\\ \left|\left|\frac{\partial\vec{x}}{\partial\nu}\right|\right| &= a\left(\sinh^2\mu+\sin^2\nu\right)^{\frac{1}{2}} = h_1\end{aligned}$$

1.5.2 Line Elements

1.5.3 The Gradient Operator

1.5.4 The Divergence Operator

1.5.5 The Curl Operator

1.5.6 The Laplacian Operator