Électromagnétisme

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Chapter 1

Electrostatics

1.1 Notion of Charge

1.1.1 Experimental observations

Electrical phenomena have been known to mankind for thousands of years. However, modern research only dates back to the 18th century. In 1785 Coulomb introduced his famous law, and by 1875 Maxwell had formulated the theory of electromagnetism.

Electromagnetic interactions, along with gravity, are the most important of the physical forces at the macroscopic scale. However, electromagnetism reveals itself directly everywhere in nature in elementary particles, atoms, molecules, biological systems, solids, and even at astrophysical scales, e.g. in stellar atmospheres.

1.1.2 Elementary particles

In our current understanding of nature at the most fundamental level, the universe is made out of particles[1], the largest fraction of which carries charge.

[1] In a more sophisticated reading: the quanta of associated fields

Particle	type	mass [kg]	charge [C=A·s]	sign
proton (p^+)	baryon	$1.678 \cdot 10^{-27}$	$1.6022 \cdot 10^{-19}$	+
neutron (n)	baryon	$1.675 \cdot 10^{-27}$	0	0
(u quark	quark	$7.5 \cdot 10^{-30}$	$\frac{2}{3}C_{p^+}$	+)
$\Delta(uuu)$	baryon	$2.196 \cdot 10^{-27}$	$2C_{p^+}$	+
$W^\pm$ boson	mediator	$1.463 \cdot 10^{-25}$	$\pm C_{p^+}$	+
electron (e^-)	lepton	$9.109 \cdot 10^{-31}$	$-1.6022 \cdot 10^{-19}$	−
myon (μ)	lepton	$1.900 \cdot 10^{-28}$	$-1.6022 \cdot 10^{-19}$	−

Table 1.1: Some (elementary and composite) particles and some of their properties; quarks do not occur in isolated form, so the mass is speculative

1.1.3 Charged bodies

An important observation is that the charge of the electron is exactly (at any measured precision) opposite the charge of the proton:

$$C_{p^+} + C_{e^-} = 0 \tag{1.1}$$

Other particles such as the μ lepton ($C_\mu = C_{e^-}$) or one of the K mesons ($C_{K^+} = C_{p^+}$) carry exactly the same charge. It has therefore been reasonable to introduce an elementary unit of charge, also called e. All macroscopic objects have integer multiples of this elementary charge

$$q = (n_+ - n_-)e \tag{1.2}$$

This quantification[2] of charge has already been introduced as early as 1910 by Millikan.

1.1.4 Conservation of charge

The Standard Model (SM) of particle physics, which currently is our most well confirmed microscopic model of the entire universe, implies that in any process (mechanical, chemical, nuclear, collisional (particles), etc.) total charge is always conserved. Examples:

[2] We carefully distinguish this notion from the "quantization of charge" which is carried out in Quantum Field Theory and employed in elementary particle physics.

- Radioactive decay:
  $n \longrightarrow p^+ + e^- + \overline{\nu}_e$
  $q = 0 \longrightarrow q = +e + (-e) + 0 = 0$

- Pair creation and annihilation:
  $e^- + e^+ \leftrightarrow 2\gamma$
  $q = -e + (+e) = 0$

- π-meson decay[3]:
  $\pi^- \longrightarrow \mu^- + \overline{\nu}_\mu$
  $q = -1 \longrightarrow q = -1$

where ν_e is the electron neutrino[4] and γ is the photon. In addition, charge does not depend on the frame of reference. Electric charge is of importance in fundamental symmetries of the universe, e.g. the celebrated $\mathcal{CPT}$ theorem.

If charge is to be conserved in a finite volume (which is a special case of charge conservation) then the entering charge must be exactly compensated by the exiting charge: such that q in $\mathcal{V}$ is conserved if

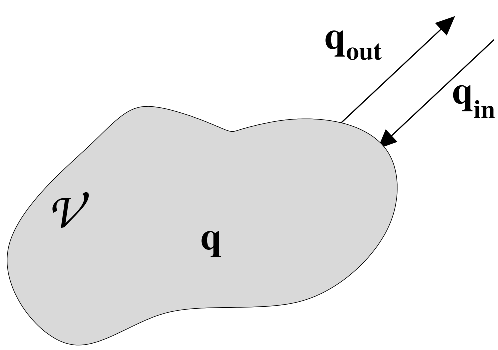


Figure 1.1: Flow of charge into and out of a delimited region $\mathcal{V}$

$$q(t_1) = q(t_2) + q_{\text{in}}(t_1 - t_2) - q_{\text{out}}(t_1 - t_2) \quad (1.3)$$

$q_{\text{in}}(t_1 - t_2) = q_{\text{out}}(t_1 - t_2)$, with discrete time instant t_i.

[3] π-mesons are two-quark states, $\pi^- \equiv d\overline{u}$
[4] The electron neutrino is a lepton of the first generation.

1.1.5 Point charges

A “point charge” is a hypothetical charge concentrated in a point (in the mathematical sense: without spatial dimension). This is an idealization that is used in the Standard Model or in atomic physics, e.g. in representing the electron, as well as in situations where the charged bodies are much smaller than the length scale of the physical problem: An example is the model of the atomic nucleus in non-relativistic atomic physics, which is typically regarded as a massive point charge $q_{\text{nuc}} = n_{p^+}e$ in obtaining the wave function of an electron, even at the atomic length scale (10^{-10} [m]).

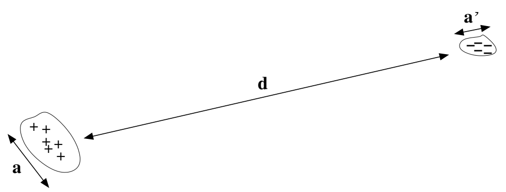


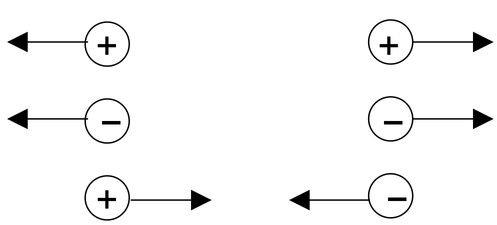
Figure 1.2: Distance between two charge distributions

Collapsing the charge distributions into point charges is warranted under the condition $\{a, a'\} <<< d$.

1.2 Coulomb’s Law

1.2.1 Positive and negative charges

Experimental evidence suggests that a charged particle exerts a force on another charged particle. This force may be attractive or repulsive:



repulsive

attractive

Figure 1.3: Attractive and repulsive forces between charged particles

1.2.2 Direction

The force between charged particles always follows a straight line connecting the particles' positions. These positions are points in isotropic space.

1.2.3 Distance dependence

The magnitude of the force between charges diminishes with the inverse square of the distance ("$\frac{1}{R^2}$" law). The validity of this law has been tested from 10^{-15} m through very large distances.

1.2.4 Formal representation of Coulomb's Law

Be two point charges q_1 and q_2 in vacuum. We define the vector $\vec{F}_{12}$

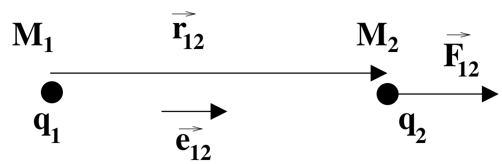


Figure 1.4: Force exerted by particle 1 on particle 2

as the force particle 1 exerts on particle 2. According to Newton's laws $\vec{F}_{12} = -\vec{F}_{21}$. We call $\vec{r}_{12}$ the distance vector connecting the points M_1 and M_2 and $\vec{e}_{12} = \frac{1}{||\vec{r}_{12}||}\vec{r}_{12}$ the unit vector in that direction.

Then Coulomb's law reads

$$\vec{F}_{12} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r_{12}^2} \vec{e}_{12} \tag{1.4}$$

with

$$\varepsilon_0 \approx 8.854 \cdot 10^{-12}\ [\mathrm{kg}^{-1}\ \mathrm{m}^{-3}\ \mathrm{s}^4\ \mathrm{A}^2] = 8.854 \cdot 10^{-12}\ [\mathrm{N}^{-1}\ \mathrm{m}^{-2}\ \mathrm{C}^2]$$

the vacuum permittivity. It is related[5] to the vacuum speed of light via the vacuum permeability μ_0 as $\varepsilon_0 = \frac{1}{\mu_0 c_0^2}$.

[5] The value for ε_0 in fact results from this equation.

In media other than the vacuum the permittivity is replaced by $\varepsilon = \varepsilon_0 \varepsilon_r$ where $\varepsilon_r > 1$ is the relative permittivity, a dimensionless parameter. Examples: in air $\varepsilon_r \approx 1.006$, in water $\varepsilon_r \approx 80$.

1.2.5 Superposition principle

Be a point charge q_0 and N other point charges $q_1, q_2, \ldots, q_N$. The

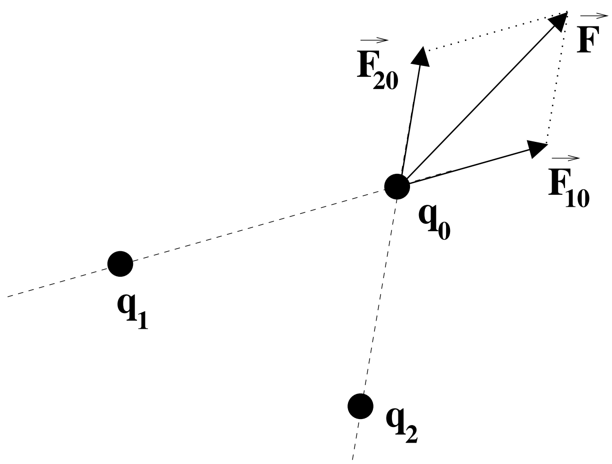


Figure 1.5: Total force exerted on particle by N other particles

total electrostatic force exerted on q_0 is then

$$\vec{F} = \sum_{i=1}^{N} \vec{F}_{i0} \tag{1.5}$$

From this superposition of forces it follows that in general

$$||\vec{F}|| \neq \sum_{i=1}^{N} ||\vec{F}_{i0}||, \tag{1.6}$$

except in the special case of aligned charges.

1.3 The Electric Field

1.3.1 Notion of Field

A field associates a quantity to every point of a region in space.

Be M a point in space, then $\Phi(M)$ is called a **scalar field**, if Φ is a scalar. An example of a scalar field is the temperature field.

$\vec{G}(M)$ is called a **vector field**, if $\vec{G}$ is a 3-component vector. Examples of vector fields are the gravitational field and the electrostatic field.

If the field is identical in every point in space, the field is called **uniform**.

If the field is constant in time for any point in space, the field is called **stationary**.

1.3.2 The Electric Field of a Point Charge

We wish to define the electric field $\vec{E}(M)$ in every point in space. For point charges, the point of departure is Coulomb's Law (Eq. 1.4) which we can re-write as

$$\vec{F}_{12} = q_2 \vec{E}(M_2). \tag{1.7}$$

Then

$$\vec{E}(M_2) = \frac{1}{4\pi\varepsilon_0} \frac{q_1}{r_{12}^2} \vec{e}_{12} \tag{1.8}$$

is the electric field at the point M_2 produced by a point charge q_1 located at a distance $||\vec{r}_{12}||$. It is sometimes useful to introduce a position vector $\vec{r}$ having its origin at the position of the point charge, in which case the expression for the electric vector field becomes

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \vec{e}_r \tag{1.9}$$

with $r = ||\vec{r}||$ and $\vec{e}_r = \frac{\vec{r}}{r}$.

One can ask the question as to whether $\vec{E}(M)$ is measurable, and how. Placing a charge q into the field and measuring the force on q introduces the problem that the initial field is changed by the presence of q. The problem is solved by using a test charge δq that is very small compared to the charges creating the field. Then, the electrostatic field is measured as

$$\vec{E} = \lim_{\delta q \to 0} \frac{\delta \vec{F}}{\delta q}. \tag{1.10}$$

1.3.3 The Superposition Principle of Fields

We now introduce the definition of the electrostatic field into the more general expression (1.5) for a spatial distribution of point charges. The electrostatic force acting on a charge q_0 at the point M_0 can thus be written as

$$\vec{F}_0 = q_0 \vec{E}(M_0) = \sum_{i=1}^{N} \vec{F}_{i0}. \tag{1.11}$$

with

$$\vec{E}(M_0) = \sum_{i=1}^{N} \frac{1}{4\pi\varepsilon_0} \frac{q_i}{r_{i0}^2} \vec{e}_{i0}. \tag{1.12}$$

It follows that since the electric field due to one of the N particles is given by $\vec{E}_{i0} = \frac{1}{4\pi\varepsilon_0} \frac{q_i}{r_{i0}^2} \vec{e}_{i0}$ the superposition principle for the field can directly be written as

$$\vec{E}_0 = \sum_{i=1}^{N} \vec{E}_{i0}. \tag{1.13}$$

The superposition principle of fields therefore follows from the superposition principle of forces on charges (which is the result of experiment).

1.3.4 Continuous Charge Distributions

A charged system may contain very many particles and/or have a very complicated structure. In such cases it can be useful to define fields, forces, etc. in terms of a continuous charge density. If the charge density is uniform, then the total charge inside a volume $\mathcal{V}$ is simply

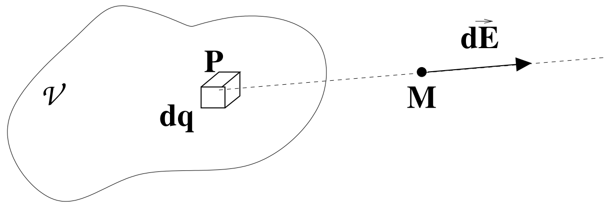


Figure 1.6: Charge distribution represented by a charge density $\rho_{\mathcal{V}}(P)$ inside a delimiting volume $\mathcal{V}$. An infinitesimal volume $d\mathcal{V}$ around a point P contains the charge dq.

given by $Q = \rho_{\mathcal{V}}\mathcal{V}$. In the general case of a non-uniform charge density, the total charge is given by the volume integral

$$Q = \iiint_{\mathcal{V}} \rho_{\mathcal{V}}\, d\mathcal{V}. \tag{1.14}$$

In order to obtain the electric field in a point M we define the infinitesimal contribution $d\vec{E}$ due to the charge dq located at point P:

$$d\vec{E}(M) = \frac{1}{4\pi\varepsilon_0}\frac{dq(P)}{||\vec{PM}||^2}\vec{e}_{PM} = \frac{1}{4\pi\varepsilon_0}\frac{\rho_{\mathcal{V}}(P)d\mathcal{V}}{||\vec{PM}||^2}\vec{e}_{PM} \tag{1.15}$$

The total electric field is therefore obtained by integrating over all $d\mathcal{V}$ located at all P inside $\mathcal{V}$:

$$\vec{E}(M) = \iiint_{\mathcal{V}} d\vec{E}(M) = \frac{1}{4\pi\varepsilon_0}\iiint_{\mathcal{V}} \frac{\rho_{\mathcal{V}}(P)\vec{e}_{PM}}{||\vec{PM}||^2} d\mathcal{V} \tag{1.16}$$

The expression simplifies to surface integrals in the case of surface charge distributions or line integrals in the case of linear charge distributions.

1.3.5 Field Lines

A field line is by definition the line that is created by tracing the direction of the vector field in each point of a topological path. In more complicated cases (e.g. the electric dipole field) this can be achieved

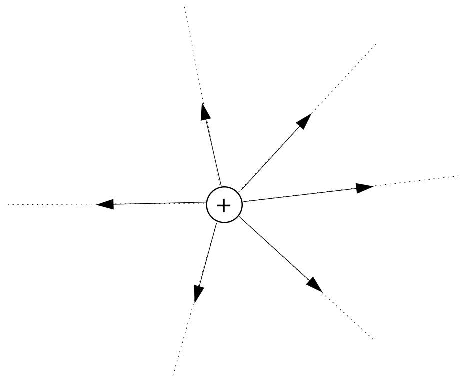


Figure 1.7: A few field lines of a positive (idealized) point charge.

The test charge is positive, by definition, so that the field lines point towards the charge if it is negative.

by discretizing the path and then letting the discrete path segments become infinitesimally small.

In practice, one chooses a point in space, calculates the electric field vector in that point, attaches the vector to that point, and uses a point along the direction of the vector as the new point. Repeating the procedure produces the field line, in a discretized form. If the distances between test points becomes infinitesimally small, the true field line is obtained.

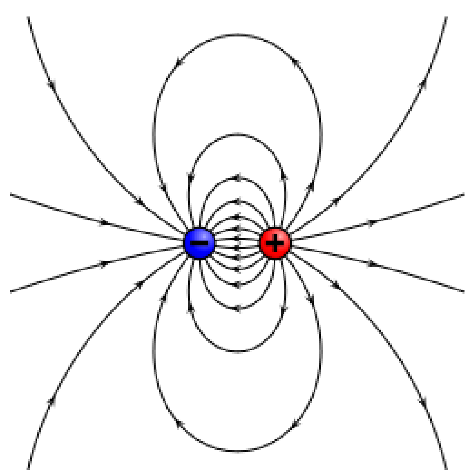


Figure 1.8: A few field lines of an electric dipole field.

1.4 The Electric Potential V

1.4.1 Line Integral

The calculus of electrostatics and -dynamics very often involves the integration along a path in coordinate space. In order to define the line integral, we first discretize this path into a finite number of points along the path $\mathcal{C}$. We can now obtain the line integral over the vector

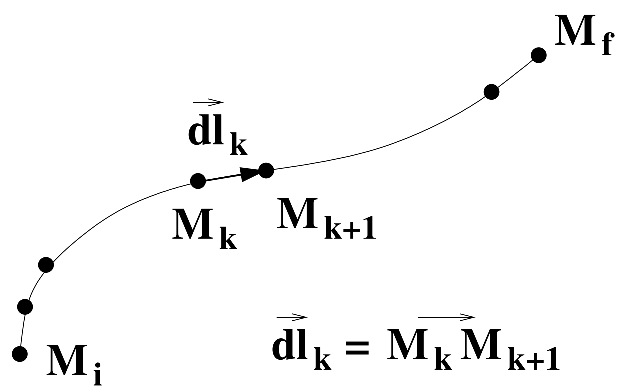


Figure 1.9: Discretized path with N connected points M_N; a vector defines the step from point M_k to point M_{k+1}.

field $\vec{G}$ by letting the number of points along the path tend to infinity, which entails that the step length becomes infinitesimally small:

$$\int_{\mathcal{C}} \vec{G} \cdot \vec{dl} := \lim_{\substack{N\to\infty \\ ||\vec{dl}_k||\to 0}} \sum_{k=1}^{N} \vec{G}(M_k) \cdot \vec{dl}_k \tag{1.17}$$

The value of the line integral may also be denoted $C_{M_i \to M_f} = \int\limits_{\mathcal{C}} \vec{G} \cdot \vec{dl}$ ("circulation"). The path may be, but does not have to be closed.

Since we are often confronted with n superposed vector fields, we note that due to the linearity of the operation of integration, we may write

$$\int_{\mathcal{C}} \sum_{i}^{n} \vec{G}_i \cdot \vec{dl} = \sum_{i}^{n} \int_{\mathcal{C}} \vec{G}_i \cdot \vec{dl} = \sum_{i}^{n} C_i \tag{1.18}$$

In the following we consider the important example of the electric field of a point charge.

1.4.2 Line integral over the Electrostatic Field of a Point Charge

The electrostatic potential of a point charge may be determined from the following general configuration (Fig. 1.10). Integration along the

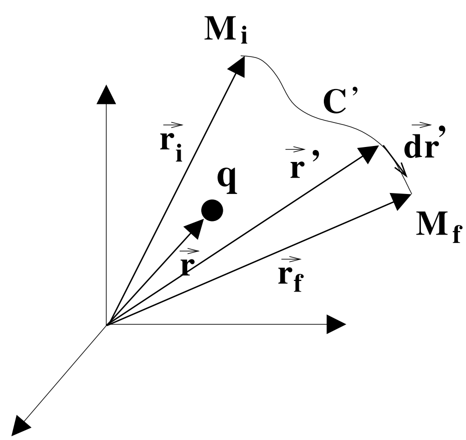


Figure 1.10: Point charge q at a general position and integration path between initial and final points.

path $\mathcal{C}'$ therefore yields the expression

$$\int_{\mathcal{C}'} \vec{E}(\vec{r}\,') \cdot \vec{dr}\,' = \frac{q}{4\pi\varepsilon_0} \int_{M_i}^{M_f} \frac{\vec{r}\,' - \vec{r}}{||\vec{r}\,' - \vec{r}||^3} \cdot \vec{dr}\,'. \tag{1.19}$$

We simplify this general expression by shifting the cartesian coordinate frame such that the charge q comes to lie at its origin ($\vec{r} = \vec{0}$). We then obtain

$$\int_{M_i}^{M_f} \vec{E}(\vec{r}) \cdot \vec{dl} = \frac{q}{4\pi\varepsilon_0} \int_{M_i}^{M_f} \frac{\vec{e}_r}{r^2} \cdot \vec{dl} \tag{1.20}$$

dropping the primes. It is now convenient to consider the spherical symmetry of the electric field of the point charge. We may therefore express the infinitesimal displacement vector in spherical polar coordinates[6]:

$$\begin{aligned}\vec{dl} &= dx\vec{e}_x + dy\vec{e}_y + dz\vec{e}_z \\ &= dl_r\vec{e}_r + dl_\vartheta\vec{e}_\vartheta + dl_\varphi\vec{e}_\varphi \qquad (1.21)\\ &= dr\vec{e}_r + rd\vartheta\vec{e}_\vartheta + r\sin\vartheta d\varphi\vec{e}_\varphi. \qquad (1.22)\end{aligned}$$

Due to $\vec{e}_r \cdot \vec{dl} = dr$ the line integral becomes

$$C_{M_i \to M_f} = \frac{q}{4\pi\varepsilon_0}\int_{r_i}^{r_f}\frac{1}{r^2}\,dr = \frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{r}\right]_{r_i}^{r_f} = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_i} - \frac{1}{r_f}\right] \qquad (1.23)$$

1.4.3 Definition of the Electrostatic Potential

We write the result of the line integral from the preceding subsection as a difference between two values of a scalar field $V(M)$ at these two points:

$$\int_{M_i}^{M_f} \vec{E}\cdot\vec{dl} = V(M_i) - V(M_f). \qquad (1.24)$$

This in turn means that the scalar field, called the electrostatic potential, can in any point of the defined space be written as

$$V(M) = \frac{q}{4\pi\varepsilon_0}\frac{1}{r(M)} + D \qquad (1.25)$$

where we have introduced an integration constant D, constant in space. The most typical (but not necessary) choice for this constant is $D = 0$ which ensures that for a point charge the potential vanishes at infinite distance:

$$\lim_{r\to\infty} V(r) = 0 \qquad (1.26)$$

[6]The spherical expression can be understood in the following way: $dl_r = dr$ is straightforward; $dl_\vartheta = rd\vartheta$ because the contour length when integrating over ϑ depends on r, and $dl_\varphi = r\sin\vartheta\, d\varphi$ because in addition to the radius the value of the polar angle plays a role in determining the contour length for the azimuthal integration.

We come to several important conclusions:

- The line integral over the electrostatic field **only** depends on the initial and end points, but **not** on the integration path chosen between these two points.

- It also immediately follows that for a closed path ($M_i = M_f$) the line integral over the electric field vanishes:

$$\int_{M_i}^{M_i} \vec{E} \cdot \vec{dl} = \oint_{\mathcal{C}} \vec{E} \cdot \vec{dl} = 0 \quad (1.27)$$

  Therefore, the electric field is called a **conservative field**. Eq. (1.27) is the first of **Maxwell's Equations**, here for the special case of electrostatics and in integral form. Its local form will be established later on.

- In case of a distribution of point charges we can use the superposition principle Eq. (1.13) and the preceding Eq. (1.27) to obtain

$$\oint_{\mathcal{C}} \vec{E} \cdot \vec{dl} = \oint_{\mathcal{C}} \sum_{i=1}^{N} \vec{E}_i \cdot \vec{dl} = \sum_{i=1}^{N} \oint_{\mathcal{C}} \vec{E}_i \cdot \vec{dl} = 0. \quad (1.28)$$

1.4.4 The Gradient Operator

We consider the change of a scalar field G along an infinitesimal displacement $\vec{dl} = dx\vec{e}_x + dy\vec{e}_y + dz\vec{e}_z$ between two points in coordinate space which can be written as a total differential

$$G(M') - G(M) = dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy + \frac{\partial G}{\partial z}dz. \quad (1.29)$$

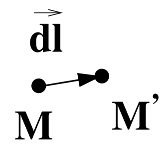


Figure 1.11: Displacement between points M and M'.

We now **define** the gradient of the scalar field G *via* the scalar product with the displacement vector:

$$dG := \vec{\text{grad}}\, G \cdot \vec{dl} \tag{1.30}$$

From this definition, the expression for the gradient of the scalar field follows as

$$\vec{\text{grad}}\, G = \frac{\partial G}{\partial x}\,\vec{e}_x + \frac{\partial G}{\partial y}\,\vec{e}_y + \frac{\partial G}{\partial z}\,\vec{e}_z \tag{1.31}$$

in cartesian coordinates. The total differential of a scalar field can thus be understood as a measure of the change of the field (given by its gradient) in the direction of a displacement.

A useful mathematical symbol for the gradient is the "nabla" operator, defined in cartesian coordinates as

$$\boldsymbol{\nabla} := \sum_{i=1}^{3} \vec{e}_i\, \frac{\partial}{\partial x_i} \tag{1.32}$$

with $\vec{e}_i$ the unit vector along the coordinate x_i.

1.4.5 The Gradient of the Electrostatic Potential

Suppose that the distance between the initial (M_i) and end point (M_f) of a line integral over a vector field $\vec{H}$ be infinitesimally small. Then the line integral becomes

$$\int_{M_i}^{M_f} \vec{H} \cdot \vec{dl} \longrightarrow \vec{H} \cdot M_i\vec{M}_f = \vec{H} \cdot \vec{dl}. \tag{1.33}$$

Using Eq. (1.30) we can therefore write

$$\vec{H} \cdot \vec{dl} = dG \tag{1.34}$$

and identify the vector field as the gradient of a scalar field:

$$\vec{H} = \vec{\mathrm{grad}}\, G \tag{1.35}$$

In the case of the electrostatic potential as scalar field, the sign of the gradient is negative in order to conform with the conventions in subsections 1.2.4 and 1.3.5. Thus

$$\vec{E} = -\vec{\mathrm{grad}}\, V \tag{1.36}$$

1.4.6 Electrostatic Potential for Continuous Charge Distributions

In case the charge distribution is continuous (see Fig. (1.6)) we may proceed by analogy to Eq. (1.15) and define a differential contribution to the electric potential as

$$dV(M) = \frac{1}{4\pi\varepsilon_0} \frac{dq}{r(M)} \tag{1.37}$$

and represent the differential charge dq by the charge density. As an example for a surface charge density σ the differential charge becomes $dq = \sigma\, dS$, with dS a surface element. The electric potential in a point M is then obtained from the integral

$$V(M) = \frac{1}{4\pi\varepsilon_0} \iint_S \frac{\sigma\, dS}{||\vec{PM}||}. \tag{1.38}$$

1.5 Symmetry of the Electrostatic Field and Potential

Symmetry plays an enormous role in physics. The properties and interactions of the constituents of the universe with respect to fundamental symmetries such as spatial parity, time reversal, charge conjugation, rotations, etc., **define** modern physical theories, such as the Standard Model of elementary particles. The search for new physics beyond the Standard Model is practically always connected to violations of such fundamental symmetries.

In the case of electrostatics, it is useful to consider the transformation properties of the field and the potential with respect to planar reflections. We understand a **symmetry transformation** as an operation that leaves the properties of a physical system invariant.

1.5.1 Electrostatic Potential

Suppose that a charge distribution ρ has a symmetry plane S, i.e., reflecting[7] the charge distribution at S yields precisely the same charge distribution, Fig. (1.12). Due to the reflection symmetry of the charge

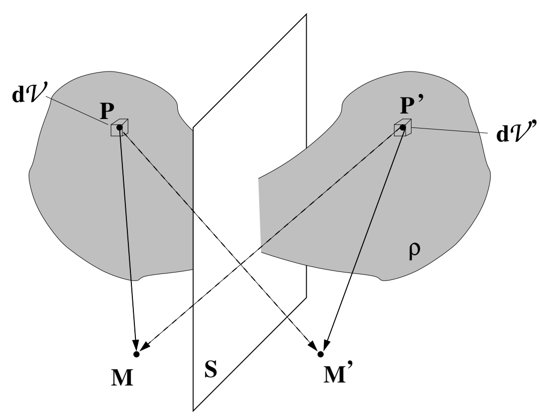


Figure 1.12: Charge distribution with a symmetry plane S.

distribution, it follows that for charge density in $d\mathcal{V} = d\mathcal{V}'$ around P (P')

$$\rho(P) = \rho(P'), \tag{1.39}$$

[7] We understand that "reflecting" means taking an elementary volume unit at one side of the symmetry plane and displacing it along a straight line perpendicular to the symmetry plane to a position at the other side of the plane which is at equal distance from the plane as the original position.

and due to $dq = \rho(P)\, d\mathcal{V}$

$$dq = dq'. \tag{1.40}$$

The elementary electrostatic potentials at points M and M' due to two symmetrically displaced elementary volumes are then using Eqs. (1.37) and (1.40)

$$\begin{aligned} dV(M) &= \frac{1}{4\pi\varepsilon_0}\left(\frac{dq}{||\vec{PM}||} + \frac{dq'}{||\vec{P'M}||}\right) = \frac{1}{4\pi\varepsilon_0}\, dq\left(\frac{1}{||\vec{PM}||} + \frac{1}{||\vec{P'M}||}\right) \\ dV(M') &= \frac{1}{4\pi\varepsilon_0}\left(\frac{dq}{||\vec{PM'}||} + \frac{dq'}{||\vec{P'M'}||}\right) = \frac{1}{4\pi\varepsilon_0}\, dq\left(\frac{1}{||\vec{PM'}||} + \frac{1}{||\vec{P'M'}||}\right) \end{aligned} \tag{1.41}$$

However, $||\vec{PM}|| = ||\vec{P'M'}||$ and $||\vec{P'M}|| = ||\vec{PM'}||$, and therefore $dV(M) = dV(M')$. Integration over $\mathcal{V}$ results in

$$V(M) = V(M'). \tag{1.42}$$

We have established an explicit symmetry relation for the electrostatic potential in the presence of a symmetry plane.

1.5.2 Electrostatic Field

From an identical reasoning we can conclude on the symmetry of the electrostatic field. From Eqs. (1.15) and (1.40)

$$\begin{aligned} d\vec{E}(M) &= \frac{1}{4\pi\varepsilon_0}\, dq\left(\frac{\vec{PM}}{||\vec{PM}||^3} + \frac{\vec{P'M}}{||\vec{P'M}||^3}\right) \\ d\vec{E}(M') &= \frac{1}{4\pi\varepsilon_0}\, dq\left(\frac{\vec{PM'}}{||\vec{PM'}||^3} + \frac{\vec{P'M'}}{||\vec{P'M'}||^3}\right). \end{aligned} \tag{1.43}$$

As before, $||\vec{PM}|| = ||\vec{P'M'}||$ and $||\vec{P'M}|| = ||\vec{PM'}||$. Concerning the vectors themselves, Fig. (1.13) shows that although they in principle are 3-dimensional, we note that the points (P, P', M, M') necessarily

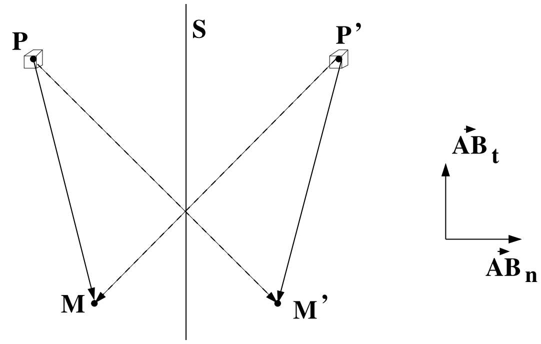


Figure 1.13: Normal and tangential components of vectors $\vec{AB}$ with respect to a symmetry plane S.

come to lie in a plane, and therefore only two components of a cartesian vector need to be considered here[8]. Introducing a normal (n) and a tangential (t) component, and noting that

$$\begin{aligned} \vec{PM}_n &= -\vec{P'M'}_n \\ \vec{PM}_t &= \vec{P'M'}_t \\ \vec{P'M}_n &= -\vec{PM'}_n \\ \vec{P'M}_t &= \vec{PM'}_t, \end{aligned} \tag{1.44}$$

we can rewrite Eq. (1.43) as

$$\begin{aligned} d\vec{E}_n(M) &= \frac{1}{4\pi\varepsilon_0} dq \left(\frac{\vec{PM}_n}{||\vec{PM}||^3} + \frac{\vec{P'M}_n}{||\vec{P'M}||^3} \right) \\ d\vec{E}_n(M') &= \frac{1}{4\pi\varepsilon_0} dq \left(-\frac{\vec{P'M}_n}{||\vec{P'M}||^3} - \frac{\vec{PM}_n}{||\vec{PM}||^3} \right) \\ d\vec{E}_t(M) &= \frac{1}{4\pi\varepsilon_0} dq \left(\frac{\vec{PM}_t}{||\vec{PM}||^3} + \frac{\vec{P'M}_t}{||\vec{P'M}||^3} \right) \\ d\vec{E}_t(M') &= \frac{1}{4\pi\varepsilon_0} dq \left(\frac{\vec{P'M}_t}{||\vec{P'M}||^3} + \frac{\vec{PM}_t}{||\vec{PM}||^3} \right) \end{aligned} \tag{1.45}$$

[8]In other words, S can be rotated freely around its normal vector such that the various vectors would only have two components with respect to a cartesian coordinate system in S.

which yields two final relations for the normal and tangential components of the electric field:

$$\boxed{\begin{aligned}\vec{E}_n(M) &= -\vec{E}_n(M') \\ \vec{E}_t(M) &= \vec{E}_t(M')\end{aligned}} \qquad (1.46)$$

1.5.3 Points in Planes of Symmetry

A very important special case of the above elucidation is comprised by points lying **in** a plane that has been identified as a symmetry plane of the charge distribution. In this case $M = M'$ and thus

$$\vec{E}_n(M) = -\vec{E}_n(M) = \vec{0}$$

leaving $\vec{E}_t(M)$ as the only non-vanishing component. This means that the electric field vector is **contained** in any symmetry plane S of a charge distribution:

$$\vec{E}(M_S) \;\parallel\; S \qquad (1.47)$$

This is a very powerful theorem that can be used to determine the direction of the electric field for many typical charge distributions.

1.5.4 Planes of Antisymmetry

Since there are two kinds of electric charges in the universe (positive and negative) the case may occur in which a charge distribution has an antisymmetry plane, A. In accord with the illustration in Fig. (1.12) this means that $\rho(P) = -\rho(P')$ and therefore $dq = -dq'$. From the above relations (1.41) and (1.43) we easily deduce

$$\boxed{\begin{aligned} V(M) &= -V(M') \\ \vec{E}_n(M) &= \vec{E}_n(M') \\ \vec{E}_t(M) &= -\vec{E}_t(M'). \quad (1.48) \end{aligned}}$$

For points lying in a plane of antisymmetry A, i.e. $M = M'$, the potential and the tangential component of the electric field vanish:

$$\begin{aligned} V(M) &= -V(M) = 0 \\ \vec{E}_t(M) &= -\vec{E}_t(M) = \vec{0} \end{aligned} \tag{1.49}$$

$\vec{E}$, therefore, is perpendicular to planes of antisymmetry:

$$\vec{E}(M_A) \perp A \tag{1.50}$$

1.6 Gauss's Theorem

This is one of the central theorems of electrostatics. It follows from geometric considerations and from the form of the electric field due to charges. We first discuss the notions of flux and solid angle and then deduce Gauss's theorem.

1.6.1 Flux of a Vector Field

1.6.1.1 Orientation of a surface

In case of a closed surface a criterion of orientation exists, since there is an inner and an outer region separated by the surface. However, for open surfaces such a criterion does not exist. It is therefore necessary to define the **orientation** of the surface, because the direction of flux through the surface must be defined. The right-hand rule[9] defines the orientation of $\vec{n}$, and therefore of the oriented surface element $\vec{n}dS$.

[9] Let your thumb point from the center of dS towards the oriented path, your index finger in the direction of the oriented path; then your middle finger defines the direction of $\vec{n}$.

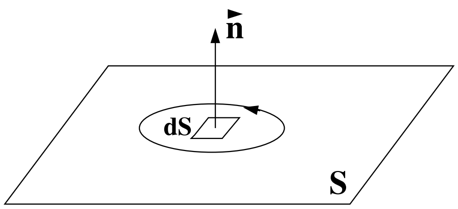


Figure 1.14: Definition of the orientation of an open surface.

- dS is a differential surface element.
- $\vec{n}$ is a normalized vector orthogonal to dS.
- Trace a closed path following the surface S around dS.
- The direction of $\vec{n}$ is given by the right-hand rule (ou “règle de tire-bouchon”).

If the surface is closed, $\vec{n}$ points outward, by convention.

1.6.1.2 Definition of the Flux of a Vector Field

The flux Φ of a vector field $\vec{G}$ through an oriented surface S is defined by

$$\Phi := \iint_{\mathcal{S}} \vec{G} \cdot \vec{n} \, dS \tag{1.51}$$

1.6.1.3 The Solid Angle

The solid angle is a generalization of the notion of the usual planar angle to a 3-dimensional context.

We first prove the following theorem for the planar angle: If the

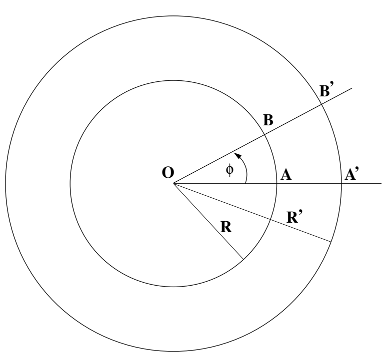


Figure 1.15: Relations of segment lengths, radius and planar angle.

One-dimensional segment theorem:

$$\frac{\overline{AB}}{R} = \frac{\overline{A'B'}}{R'} = \varphi \tag{1.52}$$

circles of Fig. 1.15 are replaced by spheres, the planar angle ϑ becomes the **solid angle**, called Ω, which is the area of a segment of a unit sphere ($R = 1$) which is centered at the angle's vertex. The solid angle

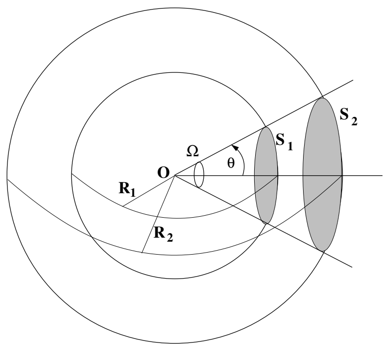


Figure 1.16: Relations of segment surfaces, radius and solid angle.

Two-dimensional segment theorem:

$$\frac{S_1}{R_1^2} = \frac{S_2}{R_2^2} = \Omega \tag{1.53}$$

is measured in the dimensionless unit *steradian*.

We note two special cases:

I) $\vartheta = \frac{\pi}{2}$. In this case one should obtain the solid angle of a demi-sphere, and indeed, $\Omega = 2\pi\left(1 - \cos\left(\frac{\pi}{2}\right)\right) = 2\pi$.

II) $\vartheta = \pi$. Here, the solid angle of the full sphere is obtained: $\Omega = 2\pi\left(1 - \cos\left(\pi\right)\right) = 4\pi$

1.6.1.4 Elementary Solid Angle

In order to generalize the use of the solid angle, we have to depart from the surface of a sphere to more complicated, general surfaces.

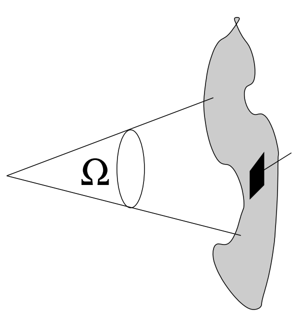


Figure 1.17: Solid angle in case of a general surface.

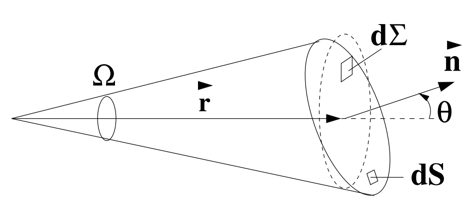


Figure 1.18: The elementary solid angle defined **via** the elementary cone.

In Fig. (1.18) $d\Sigma$ is an elementary surface segment for the surface orthogonal to $\vec{r}$. The differential solid angle can thus be written as

$$d\Omega = \frac{d\Sigma}{r^2}, \tag{1.54}$$

using the two-dimensional segment theorem, Eq. (1.53), with $r = \vec{r}\cdot\vec{e}_r$.

However, the normal vector onto the true surface of interest may be at an angle ϑ with $\vec{r}$. The two surface elements are related to each other by the scalar product of the two respective normal vectors, as

$$d\Sigma = \vec{e}_r \cdot \vec{n}\, dS \tag{1.55}$$

since in the collinear case $\Sigma = S$. In the orthogonal "limit" S becomes infinitely large (dS scales accordingly) as the scalar product[10] $\vec{e}_r \cdot \vec{n} \to 0$.

Therefore, the elementary solid angle, Eq. (1.54) can be written as

$$d\Omega = \frac{\vec{r}\cdot\vec{n}}{r^3}\, dS \tag{1.56}$$

and the surface integral yields the solid angle

$$\Omega = \iint\limits_S \frac{\vec{r}\cdot\vec{n}}{r^3}\, dS. \tag{1.57}$$

As an important example, we consider the case of a sphere. We wish to determine the solid angle under which the entire sphere appears, and we verify the earlier result:

$$\begin{aligned}\Omega &= \int_0^{\pi}\int_0^{2\pi} \frac{\vec{e}_r\cdot\vec{n}}{r^2}\, r^2 \sin\vartheta d\vartheta d\varphi \\ &= \int_0^{\pi} \sin\vartheta d\vartheta \int_0^{2\pi} d\varphi = [-\cos\vartheta]_0^{\pi}\ [\varphi]_0^{2\pi} = 4\pi \end{aligned} \tag{1.58}$$

Note that $\vec{e}_r \cdot \vec{n} = 1$ can be chosen for the spherical case.

[10] We use $\vec{e}_r \cdot \vec{n} = \cos\vartheta$ for this argument.

1.6.1.5 Multiply Intersected Closed Surface

The case may occur in which the vertex for taking the solid angle (the origin) lies **outside** of a closed surface. This situation is shown in Fig. (1.19). The two elementary solid angles can now be written, using Eq.

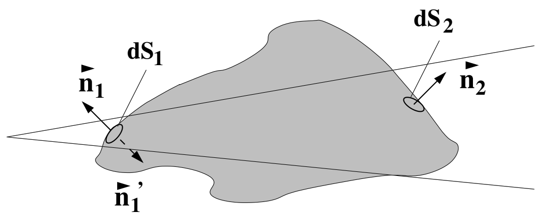


Figure 1.19: Vertex for the solid angle outside of a closed surface.

(1.56), as

$$\begin{aligned} d\Omega_1 &= \frac{\vec{r_1}\cdot\vec{n_1}}{r_1^3}\,dS_1 = -\frac{\vec{r_1}\cdot\vec{n_1'}}{r_1^3}\,dS_1 \\ d\Omega_2 &= \frac{\vec{r_2}\cdot\vec{n_2}}{r_2^3}\,dS_2 \end{aligned} \tag{1.59}$$

However, since the elementary cone is the same in either case, using theorem (1.53) the identity

$$\frac{\vec{r_1}\cdot\vec{n_1'}}{r_1^3}\,dS_1 = \frac{\vec{r_2}\cdot\vec{n_2}}{r_2^3}\,dS_2 \tag{1.60}$$

has to hold. Note that the reversed vector, $\vec{n_1'}$, has to be used here for which $\cos\vartheta \geq 0$, the angle formed by $\vec{n_1'}$ and $\vec{r}$. From this it follows that

$$d\Omega_1 + d\Omega_2 = 0 \tag{1.61}$$

for all elementary cones. Since the total solid angle with respect to the vertex is the sum of the elementary solid angles,

$$\Omega = 0 \tag{1.62}$$

for a vertex outside a closed surface. We can generalize this result: $\Omega = 0$ if the considered cone intersects the closed surface an **even** number of times.

1.6.2 Gauss's Theorem

The flux Φ of the electrostatic field of a point charge q through a surface S is

$$\Phi = \iint\limits_{\mathcal{S}} \vec{E} \cdot \vec{n}\, dS = \frac{q}{4\pi\varepsilon_0} \iint\limits_{\mathcal{S}} \frac{1}{r^3} \vec{r} \cdot \vec{n}\, dS \tag{1.63}$$

and using the solid angle from Eq. (1.57) we can write

$$\Phi = \frac{q}{4\pi\varepsilon_0} \Omega. \tag{1.64}$$

This result is reasonable, since the solid angle is a measure of the surface segment and therefore has to be related to the flux through the surface segment.

For an arbitrary ensemble of charges q_i we can therefore define a solid angle contribution Ω_i with respect to a given **closed** surface[11]. Thus,

$$\Phi = \frac{1}{4\pi\varepsilon_0} \sum_i q_i \, \Omega_i \tag{1.65}$$

and we distinguish between two cases:

$$\begin{aligned} \Omega_i &= 0 \quad \text{if } q_i \text{ is outside } S, \\ \Omega_i &= 4\pi \quad \text{if } q_i \text{ is inside } S. \end{aligned}$$

The first statement follows from the considerations in subsection 1.6.1.5. The second statement is due to Eq. (1.53) with $\vartheta = \pi$.

We may therefore write for an arbitrary ensemble of charges located **inside** a closed surface:

$$\Phi = \frac{1}{4\pi\varepsilon_0} 4\pi \sum_{i_{\text{in}}} q_i = \frac{Q_{\text{in}}}{\varepsilon_0} \tag{1.66}$$

[11] This is nothing else than applying the superposition principle of fields.

With Eq. (1.63) we finally obtain Gauss's Theorem:

$$\oiint_{\mathcal{S}} \vec{E} \cdot \vec{n}\, dS = \frac{Q_{\text{in}}}{\varepsilon_0} \tag{1.67}$$

It states that the flux of the electric field through a closed surface is proportional to the total charge located inside this closed surface. This is the second of Maxwell's Equations in electrostatics, here again in integral form. Equation (1.27) is a so-called **structure equation** for the electrostatic field, whereas Eq. (1.67) relates **field** and **source**.

In the case of a continuous charge distribution $\rho_{\mathcal{V}}$ located inside the closed surface Gauss's Theorem becomes

$$\oiint_{\mathcal{S}} \vec{E} \cdot \vec{n}\, dS = \frac{1}{\varepsilon_0} \iiint_{\mathcal{V}} \rho_{\mathcal{V}}\, d\mathcal{V}. \tag{1.68}$$

1.7 Local Form of Gauss's Theorem – Divergence

1.7.1 Definition of the Divergence of a Vector Field

The notion of divergence has been introduced in fluid dynamics. In electrodynamics (or its non-relativistic approximation electrostatics) the divergence is related to the occurrence of sources of fields.

The mathematical definition of the divergence goes as follows: Be a volume $\mathcal{V}$ delimited by a closed surface $S(\mathcal{V})$ around a point P. Then for any vector field $\vec{G}$

$$\operatorname{div} \vec{G}(P) := \lim_{\mathcal{V} \to P} \frac{1}{\mathcal{V}} \oiint_{\mathcal{S}(\mathcal{V})} \vec{G} \cdot \vec{n}\, dS \tag{1.69}$$

The divergence can thus be understood as the source density of the flux of $\vec{G}$.

How does this definition connect to the usual calculus of the divergence? As an illustration, we reduce the dimensionality of the problem. For the corresponding 2-dimensional situation we suppose that we would consider the flux through a delimiting contour $\mathcal{C}(\mathcal{S})$ of a surface $\mathcal{S}$ for which the expression becomes

$$\lim_{\mathcal{S}\to P} \frac{1}{S} \oint\limits_{\mathcal{C}(\mathcal{S})} \vec{G} \cdot \vec{n}\, dl, \tag{1.70}$$

and finally, for the 1-dimensional case introducing a cartesian coordinate x

$$\lim_{l\to P} \frac{1}{l} \left(\vec{G}(x+l) - \vec{G}(x)\right) \cdot \vec{e}_x, \tag{1.71}$$

where the integration reduces to the sum over the two end points of the interval l, and the sign is introduced due to the different orientation of the normal vector at the two end points. However, this last expression is just identical to the differential quotient (the x-component of the derivative of the vector field)

$$\lim_{l\to P} \frac{1}{l} \left(\vec{G}(x+l) - \vec{G}(x)\right) \cdot \vec{e}_x = \frac{\partial \vec{G}}{\partial x} \cdot \vec{e}_x \tag{1.72}$$

from which we infer for the 3-dimensional case the expression for the divergence:

$$\operatorname{div} \vec{G} = \frac{\partial \vec{G}}{\partial x} \cdot \vec{e}_x + \frac{\partial \vec{G}}{\partial y} \cdot \vec{e}_y + \frac{\partial \vec{G}}{\partial z} \cdot \vec{e}_z \tag{1.73}$$

Very often the symbol $\boldsymbol{\nabla}$ (“nabla”) is used to denote the divergence (as well as for the gradient in Eq. (1.31)), which in cartesian coordinates can also be written as

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\vec{G} &= \left(\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\,\vec{e}_i\right)\cdot\left(\sum_{j=1}^{3}G_j\,\vec{e}_j\right) \\
&= \sum_{i,j=1}^{3}\vec{e}_i\cdot\vec{e}_j\,\frac{\partial}{\partial x_i}\,G_j \\
&= \sum_{i,j=1}^{3}\delta_{ij}\,\frac{\partial G_j}{\partial x_i} \\
&= \sum_{i=1}^{3}\frac{\partial G_i}{\partial x_i}
\end{aligned} \tag{1.74}$$

Here we use the δ ("Kronecker") symbol, defined as

$$\delta_{ij} = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases} \qquad \forall i, j \in \mathbb{N} \tag{1.75}$$

1.7.2 Gauss's Theorem in Local Form

We now consider the special case of the electrostatic field, for which the general divergence definition Eq. (1.69) becomes

$$\begin{aligned}
\mathrm{div}\vec{E}(P) &:= \lim_{\mathcal{V}\to P}\frac{1}{\mathcal{V}}\oiint_{\mathcal{S}(\mathcal{V})}\vec{E}\cdot\vec{n}\,dS \\
&= \lim_{\mathcal{V}\to P}\frac{1}{\mathcal{V}}\,\frac{Q_{\mathcal{V}}}{\varepsilon_0},
\end{aligned} \tag{1.76}$$

where Gauss's Theorem, Eq. (1.67), has been used with respect to the volume $\mathcal{V}$. Since $\frac{Q_{\mathcal{V}}}{\mathcal{V}}$ is just a charge density for the considered volume, we may reformulate:

$$\mathrm{div}\vec{E}(P) = \lim_{\mathcal{V}\to P}\frac{\rho_{\mathcal{V}}}{\varepsilon_0} = \frac{\rho(P)}{\varepsilon_0} \tag{1.77}$$

Thus, for any point P, we obtain Gauss's Theorem in local form as

$$\text{div}\vec{E} = \frac{\rho}{\varepsilon_0} \qquad (1.78)$$

It is important to understand that this is a **local** expression, relating the charge density in a given point to the divergence of the field in that given point. The equivalent integral form of Gauss's Theorem is used when surface and volume of a given problem are well defined, whereas the local form is typically used in a more general context.

1.7.3 Point charges and Dirac's "delta function"

Let us apply Gauss's Theorem in local form, Eq. (1.78), to a seemingly very simple case, that of a point charge q. If placed at the origin, we may write the electrostatic field due to q as

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\frac{\vec{e}_r}{r^2} \qquad (1.79)$$

using Eq. (1.9). In spherical polar coordinates the radial part of the divergence operator applied to a vector field $\vec{G}$ is $\text{div}\vec{G} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 G_r\right)$, and so we obtain

$$\text{div}\vec{E} = \boldsymbol{\nabla}\cdot\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{r^2}\right) = 0. \qquad (1.80)$$

If we now integrate Gauss's Theorem over all space, using the preceding result for the divergence,

$$\iiint_{\mathcal{V}} \text{div}\vec{E}\, d\mathcal{V} = \frac{1}{\varepsilon_0}\iiint_{\mathcal{V}} \rho\, d\mathcal{V} \qquad (1.81)$$

we obtain a startling result: Since the integrand is zero, the left-hand side of Eq. (1.81) is zero, too, which means that upon performing the integration on the right-hand side, we should also obtain zero. However, basic physics demands that $\iiint_{\mathcal{V}} \rho\, d\mathcal{V} = q$, (since the point charge has to be somewhere in space) and not zero! We face a contradiction. What is going on?

The problem is related to the fact that Eq. (1.79) is incomplete. It defines the electric field of a point charge everywhere in space, except at the position of the charge, where it is undefined. And since Eq. (1.78) is a local expression, it yields the correct divergence everywhere in space, except at the position of the charge, which however becomes decisive here. This means that for consistency, the space U over which Eq. (1.78) is defined has to exclude the origin O in the case of a point charge:

$$\mathrm{div}\vec{E} = \frac{\rho}{\varepsilon_0} \qquad\qquad U\backslash\{O\} \tag{1.82}$$

In other words, due to the locality, the charge density is only defined in those points P where the electrostatic field is defined. A charge density is therefore required which describes a point charge when integrated over **all** space[12]. The solution was proposed by P.A.M. Dirac. What is required is a "function" that is zero everywhere in space, except at a given point, and that the integral over this function yields 1. This is Dirac's "delta function"[13] $\delta(x)$ which is defined *via* its integral as

$$\int\limits_{-\infty}^{+\infty} \delta(x)\, dx = 1 \tag{1.83}$$

If we furthermore define a 3-dimensional version of $\delta(x)$ as

$$\delta^{(3)}(\vec{r}) := \delta(x)\,\delta(y)\,\delta(z) \tag{1.84}$$

then the charge density for a point charge can be written as

$$\rho_q = q\,\delta^{(3)}(\vec{r}) \tag{1.85}$$

[12] Then Gauss's Theorem can be applied over all space, including the origin. We must, however, refrain from using the standard expression for the electric field of the point charge, and instead work with the divergence of $\vec{E}$ as such.

[13] $\delta(x)$ is not strictly a function, but a so-called distribution which is defined in more detail in the mathematical literature. A mnemonic is to picture $\delta(x)$ as a distribution over the integration axis with a mean width the limit of which is taken to tend to zero, however, retaining the value 1 for the integral over $\delta(x)$. It can be considered as the continuous analog of the discrete Kronecker symbol.

and the spatial integral over this charge density yields q. Therefore, the consistency of the integration in Eq. (1.81) has been regained when the divergence of the electric field is expressed in terms of a charge density for a point charge.

To summarize, in applying the local form of Gauss's Theorem to the case of distributions of point charges, care has to be taken. In those cases, however, the integral form in Eq. (1.67) is still applicable without difficulties.

1.7.4 Theorem of Gauss and Ostrogradsky (Divergence Theorem)

Eqs. (1.68) and (1.81) can be directly combined to yield

$$\iiint_{\mathcal{V}} \mathrm{div}\vec{E}\, d\mathcal{V} = \oint\!\!\oint_{S(\mathcal{V})} \vec{E} \cdot \vec{n}\, dS \tag{1.86}$$

with $S(\mathcal{V})$ the surface delimiting the integration volume, which is known as Gauss-Ostrogradsky Theorem or Divergence Theorem. It means that a property of the vector field **inside** the volume is related to the flux of the vector field traversing the delimiting surface.

The above considerations allow for a generalization of this theorem to **any vector field** $\vec{G}$, not just the electrostatic field.

$$\boxed{\iiint_{\mathcal{V}} \mathrm{div}\vec{G}\, d\mathcal{V} = \oint\!\!\oint_{S(\mathcal{V})} \vec{G} \cdot \vec{n}\, dS \qquad (1.87)}$$

1.8 Local Equations for the Electrostatic Potential

1.8.1 Poisson's Equation

Replacing in Gauss's Theorem in local form, Eq. (1.78) the electric field by the local electrostatic field structure equation $\vec{E} = -\overrightarrow{\mathrm{grad}}\, V$

(Eq. (1.36)), we obtain

$$\operatorname{div}\left(\vec{\operatorname{grad}}\, V\right) = -\frac{\rho}{\varepsilon_0}. \tag{1.88}$$

Using considerations from vector analysis, in particular Eq. (1.74), this equation may be reformulated as

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\, V) \;=\; \Delta_x\, V(\vec{x}\,) \tag{1.89}$$

where we have introduced the Laplaceoperator (or Laplacian)

$$\Delta_x := \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}. \tag{1.90}$$

We thus obtain Poisson's equation

$$\boxed{\Delta\, V = -\frac{\rho}{\varepsilon_0} \qquad (1.91)}$$

which evidently is a local equation for the electrostatic potential in the presence of charges, represented by the charge density ρ.

For a region of space without charges, Poisson's equation reduces to Laplace's equation

$$\Delta\, V = 0. \tag{1.92}$$

These two latter equations are sometimes used to solve specific problems. As an example, we mention the elementary case of a point charge. Then, Eq. (1.92) is valid for all of space, except at the position of the charge, where we have

$$\begin{aligned} \Delta \frac{q}{4\pi\varepsilon_0\, r} \;&=\; -\frac{q\, \delta^{(3)}(\vec{r})}{\varepsilon_0} \\ \Delta \frac{1}{r} \;&=\; -4\pi\, \delta^{(3)}(\vec{r}) \end{aligned} \tag{1.93}$$

This means that it is possible to represent the action of the Laplacian onto a function at a point where this function is undefined. This is a special type of differential equation, the solutions[14] of which are called **Green's functions**.

Eqs. (1.91) and (1.92) play an important role in atomic physics and Quantum Electrodynamics, but also for problems with continuous charge distributions.

1.9 Electrostatic Multipole Expansion

In this section we wish to address a central aspect of general charge distributions which plays a role in practically any microscopic context, from biochemical and chemical systems down to atoms and elementary particle theory. Suppose that an arbitrary charge distribution is located in a volume, the length scale of which is small compared to the distance from a point P of reference (such a charge distribution could for instance be given by the quarks in an atomic nucleus, the electrons in an atom, or a cluster of water molecules in a solute).

1.9.1 General Expression for the Electrostatic Potential

The electrostatic potential in a point P, at a distance $||\vec{r}||$ from the origin, for such an arbitrary charge distribution is

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{R_i}, \qquad (1.94)$$

where we have set $V(\infty) = 0$ since no charges are supposed to be located at large distance. We now rewrite the distance between P and

[14] Green's function of the Laplacian is, therefore, the function $\frac{1}{r}$.

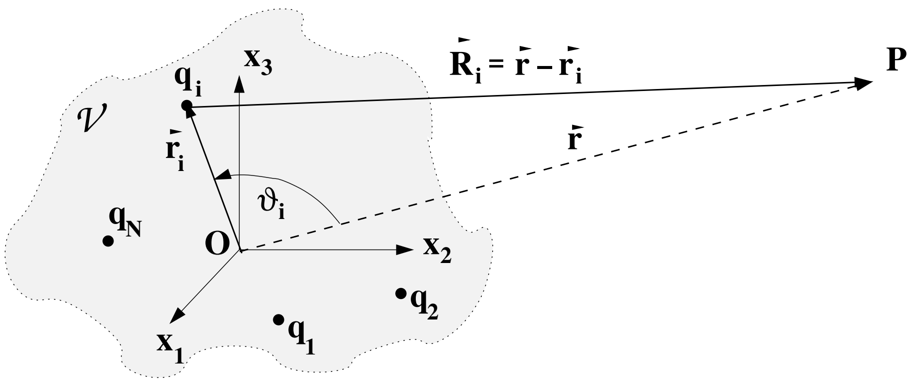


Figure 1.20: Charge distribution seen from a point P which is farther from the origin than any of the charges inside the respective region.

the ith charge as

$$\begin{aligned} R_i = ||\vec{R}_i|| &= \sqrt{(\vec{r} - \vec{r}_i)^2} \\ = \left(\vec{r}^{\,2} + \vec{r}_i^{\,2} - 2\vec{r} \cdot \vec{r}_i\right)^{\frac{1}{2}} &= \left(\vec{r}^{\,2} + \vec{r}_i^{\,2} - 2rr_i \cos\vartheta_i\right)^{\frac{1}{2}} \end{aligned} \tag{1.95}$$

with $||\vec{r}_j|| = r_j$. We take a closer look at

$$\begin{aligned} \frac{1}{R_i} &= \frac{1}{r\left[1 + \left(\frac{r_i}{r}\right)^2 - 2\frac{r_i}{r}\,\cos\vartheta_i\right]^{\frac{1}{2}}} = \frac{1}{r\,(1+t)^{\frac{1}{2}}} \\ &= \frac{1}{r}\,(1+t)^{-\frac{1}{2}} \end{aligned} \tag{1.96}$$

with the definition $t := \left(\frac{r_i}{r}\right)^2 - 2\frac{r_i}{r}\,\cos\vartheta_i$. Since under our present conditions for the multipole expansion as in Fig. (1.20)

$$\frac{r_i}{r} < 1 \qquad \forall i \tag{1.97}$$

$t > -1$ is always valid, so the square root in Eq. (1.96) never becomes imaginary. This allows us to Taylor expand as follows:

$$(1+t)^{-\frac{1}{2}} = 1 - \frac{1}{2}\,t + \frac{3}{8}\,t^2 - \frac{5}{16}\,t^3 + \ldots \tag{1.98}$$

Then Eq. (1.96) is rewritten as

$$\begin{aligned}\frac{1}{R_i} &= \frac{1}{r}\left\{1+\frac{1}{2}\left[2\frac{r_i}{r}\cos\vartheta_i-\left(\frac{r_i}{r}\right)^2\right]+\frac{3}{8}\left[-2\frac{r_i}{r}\cos\vartheta_i+\left(\frac{r_i}{r}\right)^2\right]^2\right.\\ &\qquad\left.-\frac{5}{16}\left[-2\frac{r_i}{r}\cos\vartheta_i+\left(\frac{r_i}{r}\right)^2\right]^3+\ldots\right\}\\ &= \frac{1}{r}\left\{1+\frac{r_i}{r}\cos\vartheta_i-\frac{1}{2}\left(\frac{r_i}{r}\right)^2+\frac{3}{2}\left(\frac{r_i}{r}\right)^2(\cos\vartheta_i)^2\right.\\ &\qquad\left.+\mathcal{O}\left[\left(\frac{r_i}{r}\right)^3\right]\right\}. \qquad (1.99)\end{aligned}$$

Since $\frac{r_i}{r}<1$ we know that terms of order n in $\mathcal{O}\left[\left(\frac{r_i}{r}\right)^n\right]$ become less and less important as n increases. We therefore in the following retain only terms up to $\mathcal{O}\left[\left(\frac{r_i}{r}\right)^2\right]$, remembering that the series is an infinite expansion. Thus,

$$\frac{1}{R_i}\approx\frac{1}{r}\left\{1+\frac{r_i}{r}\cos\vartheta_i+\frac{1}{2}\left(\frac{r_i}{r}\right)^2\left[3(\cos\vartheta_i)^2-1\right]\right\}. \qquad (1.100)$$

Inserting Eq. (1.100) into the expression for the electrostatic potential, Eq. (1.94), yields

$$\begin{aligned}V(\vec{r}) &= \frac{1}{4\pi\varepsilon_0 r}\sum_{i=1}^{N}q_i\\ &\quad+\frac{1}{4\pi\varepsilon_0 r^2}\sum_{i=1}^{N}q_i r_i\cos\vartheta_i\\ &\quad+\frac{1}{4\pi\varepsilon_0 r^3}\sum_{i=1}^{N}\frac{q_i r_i^2}{2}\left[3(\cos\vartheta_i)^2-1\right]\\ &\quad+\ldots \qquad (1.101)\end{aligned}$$

It is more than an interesting mathematical feature that this expression can be written in terms of a set of polynomials that has been introduced by Legendre to solve his differential equation[15]. We define Legendre's polynomials by the so-called formula of Rodriguez:

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} \left(x^2 - 1\right)^\ell \tag{1.102}$$

With $x = \cos\vartheta_j$ for the jth particle, use of Eq. (1.102) and reorganizing the terms in Eq. (1.101)

$$\begin{aligned} V(\vec{r}) &= \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} q_i \left[\frac{1}{r} + \frac{1}{r^2} r_i \cos\vartheta_i + \frac{1}{r^3} r_i^2 \left(3\left(\cos\vartheta_i\right)^2 - 1\right) + \dots\right] \\ &= \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} q_i \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} r_i^\ell P_\ell(\cos\vartheta_i) \end{aligned} \tag{1.103}$$

we finally obtain a closed expression for the electrostatic potential in terms of the introduced expansions:

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \sum_{i=1}^{N} q_i r_i^\ell P_\ell(\cos\vartheta_i) \quad (1.104)}$$

1.9.2 Individual Multipole Terms

The general electrostatic multipole expansion allows us to view a charge distribution as the sum of individual multipole terms, each with distinct characteristics. We will now analyze these terms one by one.

[15] A side remark is that Legendre's differential equation occurs when solving Laplace's equation (1.92) in spherical polar coordinates. Since this is an equation for the electrostatic potential, a direct link is established. Legendre polynomials play an essential role in the angular solutions of problems with spherical symmetry, pivotal in atomic physics.

1.9.2.1 Electric Monopole Moment

The first term in the sum over ℓ in Eq. (1.104) with $\ell = 0$ yields

$$V_{\ell=0}(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r} \sum_{i=1}^{N} q_i \tag{1.105}$$

and with the introduction of the total charge

$$\boxed{Q = \sum_{i=1}^{N} q_i \qquad (1.106)}$$

also referred to as electric **monopole moment**, we may write this term as

$$V_M(\vec{r}) = \frac{Q}{4\pi\varepsilon_0 r}. \tag{1.107}$$

In accord with its origin in electric point charges and being independent of the position of the individual charges, the term is called Monopole (M) term. At very long distance r this term will be dominant for a non-neutral distribution, because in the limit the charge distribution will be independent of its internal structure and resemble the form of a point charge Q.

For general, i.e. also continuous charge distributions, the monopole moment contribution to the potential will be written as

$$V_M(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r} \iiint\limits_{\mathcal{V}} \rho(\vec{r}\,')\, d\mathcal{V}' \tag{1.108}$$

where $\iiint\limits_{\mathcal{V}} \rho(\vec{r}\,')\, d\mathcal{V}'$ represents the monopole moment. Note that this expression is also useful for point charges with the introduction of Dirac's delta function for the charge density ρ (see subsection 1.7.3).

1.9.2.2 Electric Dipole Moment

The second term in the sum over ℓ in Eq. (1.104) with $\ell = 1$ yields

$$V_{\ell=1}(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r^2} \sum_{i=1}^{N} q_i\, r_i\, \cos\vartheta_i. \tag{1.109}$$

We wish to rewrite this into a more convenient form. Since

$$\cos\vartheta_i = \frac{\vec{r}\cdot\vec{r}_i}{r\, r_i} = \vec{e}_r \cdot \frac{\vec{r}_i}{r_i} \tag{1.110}$$

Eq. (1.109) becomes

$$V_{\ell=1}(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r^2} \sum_{i=1}^{N} q_i\, r_i\, \vec{e}_r \cdot \frac{\vec{r}_i}{r_i} = \frac{1}{4\pi\varepsilon_0 r^2}\, \vec{e}_r \cdot \left(\sum_{i=1}^{N} q_i\, \vec{r}_i \right) \tag{1.111}$$

The quantity in the parenthesis is evidently an intrinsic property of the charge distribution and independent of the point P. It is called the **electric dipole moment** which is defined as

$$\boxed{\vec{p} := \sum_{i=1}^{N} q_i\, \vec{r}_i. \qquad (1.112)}$$

With the introduction of the dipole moment the dipole term can be written as

$$V_D(\vec{r}) = \frac{\vec{e}_r \cdot \vec{p}}{4\pi\varepsilon_0 r^2}. \tag{1.113}$$

An interesting aspect of this term is that it becomes the dominant feature of a charge distribution at long distance when the system is electrically neutral ($Q = 0$).

Finally, for a general charge distribution, we write the electric dipole moment as

$$\vec{p} := \iiint_{\mathcal{V}'} \rho(\vec{r}\,')\, \vec{r}\,'\, d\mathcal{V}'. \tag{1.114}$$

1.9.2.3 Higher Multipole Moments

With $\ell = 2$, Eq. (1.104) yields the electric quadrupole moment, with $\ell = 3$ the electric octupole moment, etc. We can thus regard the electrostatic potential as a sum over individual multipole contributions, each containing a respective multipole moment:

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} V_\ell(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left\{\frac{Q}{r} + \frac{\vec{e}_r \cdot \vec{p}}{r^2} + \dots\right\} \tag{1.115}$$

1.10 Electrostatic Energy

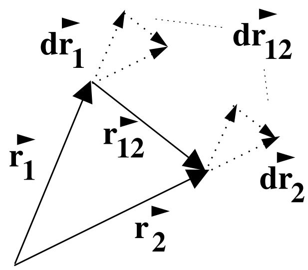
We wish to obtain an expression for the energy in electrostatics. Since we so far are not considering dynamics of particles, this will be a potential energy due to electrostatic forces.

1.10.1 Elementary Definition

Let us consider the elementary work δW two point charges q_1 and q_2 carry out along infinitesimal (virtual) displacements $d\vec{r}_1$ and $d\vec{r}_2$. With the definition of forces from subsection 1.2.4 this work can be written as

$$\begin{aligned} \delta W &= \vec{F}_{21} \cdot d\vec{r}_1 + \vec{F}_{12} \cdot d\vec{r}_2 \\ &= \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}^3}\left(\vec{r}_{21} \cdot d\vec{r}_1 + \vec{r}_{12} \cdot d\vec{r}_2\right) \\ &= \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}^3}\,\vec{r}_{12} \cdot \left(d\vec{r}_2 - d\vec{r}_1\right) \end{aligned} \tag{1.116}$$

Therefore,



The elementary displacements on individual charges combine to a displacement along the line connecting the point charges.

Figure 1.21: Elementary displacement and direction for elementary work on point charges

$$\begin{aligned}\delta W &= \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}^3}\,\vec{r}_{12}\cdot d\vec{r}_{12} \\ &= \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}^3}\,r_{12}\,dr_{12} \\ &= \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}^2}\,dr_{12} := -d\varepsilon_e \end{aligned} \tag{1.117}$$

where the elementary electrostatic energy $d\varepsilon_e$ has been defined. The electrostatic potential energy can thus be written as

$$\varepsilon_e = \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}} + \text{Cste.} \tag{1.118}$$

for two point charges q_1 and q_2. The constant is usually set to zero, which means that the energy becomes zero when the charges are infinitely separated.

It is straightforward to generalize this result to an arbitrary number N of point charges. Limiting the second summation in order to avoid double counting of terms, we obtain

$$\boxed{\varepsilon_e = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\sum_{j>i}^{N}\frac{q_i q_j}{r_{ij}} \qquad (1.119)}$$

It is also instructive to relate the electrostatic potential energy to the electrostatic potential. If we define $V(i) = \sum_{j=1}^{N}\frac{q_j}{4\pi\varepsilon_0 r_{ij}}$ as the potential at the location of a charge q_i due to N other point charges, then we obtain

$$\varepsilon_e = \frac{1}{2} \sum_{i=1}^{N} q_i V(i). \tag{1.120}$$

The prefactor $\frac{1}{2}$ compensates for the double counting of terms introduced through two unrestricted summations.

Finally, for continuous charge distributions, the electrostatic energy becomes

$$\varepsilon_e = \frac{1}{2} \iiint_{\mathcal{V}} \rho\, V\, d\mathcal{V} \tag{1.121}$$

where ρ is a local volume charge density and V is a potential function. In case of a surface charge density σ the expression becomes

$$\varepsilon_e = \frac{1}{2} \iint_{S} \sigma\, V\, dS. \tag{1.122}$$

1.10.2 Electrostatic Energy in Terms of the Electrostatic Field

We start out from Eq. (1.121). Due to Gauss's Theorem in local form, Eq. (1.78), the volume charge density can be expressed as $\rho = \varepsilon_0\, \boldsymbol{\nabla} \cdot \vec{E}$, and so the electrostatic energy

$$\varepsilon_e = \frac{\varepsilon_0}{2} \iiint_{\mathcal{V}} V \left(\boldsymbol{\nabla} \cdot \vec{E} \right) d\mathcal{V} \tag{1.123}$$

Since due to Eq. (4) in the appendix $\boldsymbol{\nabla} \cdot \left(V \vec{E} \right) = V \left(\boldsymbol{\nabla} \cdot \vec{E} \right) + \vec{E} \cdot \boldsymbol{\nabla} V$ and $\vec{E} = -\boldsymbol{\nabla} V$, Eq. (1.123) becomes

$$\varepsilon_e = \frac{\varepsilon_0}{2} \left[\iiint_{\mathcal{V}} \vec{E}^{\,2}\, d\mathcal{V} + \iiint_{\mathcal{V}} \boldsymbol{\nabla} \cdot \left(V \vec{E} \right) d\mathcal{V} \right] \tag{1.124}$$

With the help of the divergence theorem, Eq. (1.86), the second term is rewritten as

$$\iiint_{\mathcal{V}} \boldsymbol{\nabla} \cdot \left(V\vec{E}\right) d\mathcal{V} = \oiint_{S(\mathcal{V})} \left(V\vec{E}\right) \cdot \vec{n}\, dS \tag{1.125}$$

We remember that for determining the electrostatic energy, we are integrating over **all space**. This means that for any relatistic physical case, the enclosing surface can be chosen so large, that the charge distribution inside the volume appears point-like. Then, since $V \propto \frac{1}{r}$, $E \propto \frac{1}{r^2}$, and $dS \propto r^2$,

$$\lim_{r\to\infty} \oiint_{S(\mathcal{V})} \left(V\vec{E}\right) \cdot \vec{n}\, dS \longrightarrow 0 \tag{1.126}$$

and so

$$\boxed{\varepsilon_e = \frac{\varepsilon_0}{2} \iiint_{\mathcal{V}} \vec{E}^{\,2}\, d\mathcal{V}. \qquad (1.127)}$$

This is a very useful and widely used expression for the electrostatic energy, solely in terms of the electrostatic field generated by the present charges.

Chapter 2

Electric Currents

Magnetism originates from charges in motion. Before introducing the concepts of magnetostatics, we must take a closer look at the basics of charged currents.

2.1 Electrokinetics

2.1.1 Volume Charge Density

Since charged bodies are ultimately always made up of point charges, we can represent the (mean) charge density in a finite region $\mathcal{V}$ of space as

$$\rho = \frac{1}{\mathcal{V}} Q_{\mathcal{V}}, \tag{2.1}$$

where $Q_{\mathcal{V}}$ is the total charge inside $\mathcal{V}$. Supposing that the detailed distribution of point charges is not relevant for a given problem, and we content ourselves with a continuous charge distribution, then we can take the limit for a finite volume tending towards a point P and obtain

$$\rho(P) = \lim_{\Delta\mathcal{V} \to P} \frac{1}{\Delta\mathcal{V}} Q_{\Delta\mathcal{V}} \tag{2.2}$$

Eq. (2.2) supposes an "ideal" world where the point charges q_i can take on any value, i.e., they are not quantized. In the real physical world this definition naturally leads to the Dirac delta function, since then the charge density becomes infinite at points where point charges are located, and zero otherwise. In the case of a distribution of point charges we can therefore generalize Eq. (1.85) to n point charges distributed at points $\vec{r}_i$ over space and write

$$\rho(\vec{r}) = \sum_{i=1}^{n} q_i \, \delta^{(3)}(\vec{r} - \vec{r}_i) \tag{2.3}$$

where $\vec{r}$ is a position vector relative to an origin.

2.1.2 Charged Volume Current

We can write an elementary charged electric current as the charge of a particle multiplied by a velocity vector, $q\vec{v}$. The supposed continuous situation at the heart of Eq. (2.2) then allows us to write the **charged current density** as

$$\vec{J}(P) = \rho(P)\vec{v}(P) \tag{2.4}$$

where we understand $\vec{v}(P)$ as the velocity of charge in point P. From this we can infer a local expression for the charged current density in terms of the charge density at a point in space:

$$\boxed{\vec{J} = \rho\vec{v} \qquad (2.5)}$$

Note that $\vec{J}$ can be considered as a vector field. This definition of charged current density is of practical value in applications involving conductors and similar macroscopic objects.

In analogy to Eq. (2.3) and subsection 2.1.2 the charged current density for moving point charges is written as

$$\vec{J}(\vec{r}) = \sum_{j=1}^{n} q_j \, \vec{v}_j \, \delta^{(3)}(\vec{r} - \vec{r}_j) \tag{2.6}$$

The physical dimension of J is $\dim[J] = \frac{QL}{L^3T} = \frac{Q}{L^2T}$.

2.1.3 Intensity of the Electric Current

Suppose that $\vec{J}(P)$ is known for a region of space containing the points P. Then a differential intensity of electric current traversing an arbitrary differential surface in the region is defined as

$$dI(P) := \vec{J}(P) \cdot \vec{n}\, dS(P) \tag{2.7}$$

from which the **electric current intensity** I traversing a given surface is obtained by integration:

$$I = \iint_{\mathcal{S}} \vec{J} \cdot \vec{n}\, dS \tag{2.8}$$

Note that the surface $\mathcal{S}$ can be, but must not necessarily be a closed surface.

The physical dimension of I is $\dim[I] = \frac{Q}{T}$.

2.2 Conservation of Charge

2.2.1 Continuity Equation of Electromagnetism

Be a closed surface containing electric charge, Fig. (2.1)

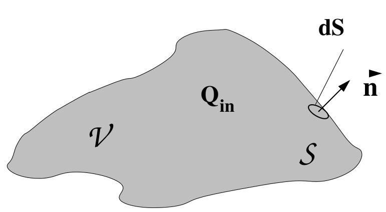


Figure 2.1: Charge inside a delimited volume.

The current intensity for charge flowing out of the volume is

$$I = -\frac{dQ_{\text{in}}}{dt} \tag{2.9}$$

which describes the time-dependent change of total charge inside the region, considering $Q_{\text{in}} = Q_{\text{in}}(t)$ as a function of time.

Using Eqs. (1.14) and (2.8) it follows that

$$\frac{d}{dt} \iiint_{\mathcal{V}} \rho_{\mathcal{V}}(\vec{r}, t)\, d\mathcal{V} = -\iint_{\mathcal{S}} \vec{J} \cdot \vec{n}\, dS. \tag{2.10}$$

where $\rho_{\mathcal{V}} = \rho_{\mathcal{V}}(\vec{r}, t)$ is considered as a time-dependent scalar field. Since the theorem of Gauss-Ostrogradsky, Eq. (1.86), is valid for **any** vector field, we may write

$$\frac{d}{dt} \iiint\limits_{\mathcal{V}} \rho_{\mathcal{V}}(\vec{r}, t)\, d\mathcal{V} = - \iiint\limits_{\mathcal{V}} \operatorname{div}\vec{J}\, d\mathcal{V}, \tag{2.11}$$

Since

$$\frac{d}{dt} \iiint\limits_{\mathcal{V}} \rho_{\mathcal{V}}(\vec{r}, t)\, d\mathcal{V} = \iiint\limits_{\mathcal{V}} \frac{\partial \rho_{\mathcal{V}}(\vec{r}, t)}{\partial t}\, d\mathcal{V},$$

Eq. (2.11) becomes

$$\iiint\limits_{\mathcal{V}} \frac{\partial \rho_{\mathcal{V}}(\vec{r}, t)}{\partial t}\, d\mathcal{V} = - \iiint\limits_{\mathcal{V}} \operatorname{div}\vec{J}\, d\mathcal{V}. \tag{2.12}$$

Eq. (2.12) holds if the integrands on either side are identical. Therefore,

$$\boxed{\frac{\partial}{\partial t} \rho_{\mathcal{V}}(\vec{r}, t) = -\operatorname{div}\vec{J}(\vec{r}, t) \qquad (2.13)}$$

which is the so-called **continuity equation** of electromagnetism. Note that in the general case of a non-linear time-dependence of the charge density inside the volume, the charged current density is also a function of time.

The continuity equation is of fundamental importance. Its interpretation is gained if we view the divergence of a vector field as related to sources of this vector field, here sources of current. Then it becomes clear that if there are (local) sources of current, the local charge density cannot be constant in time and they are related to the temporal change of the charge density. The continuity equation can therefore also be regarded as a manifestation of the conservation of charge.

2.2.2 Stationary Regime

Electrostatics as it is presented in the first chapter is an entirely time-independent theory, concerned with static configurations of electric charges. From the considerations in the second chapter it becomes clear that the time variable begins to play a role when electric currents are considered. We may, however, define a regime where there is no explicit time-dependence, even in the presence of currents. This **stationary regime** is defined by the condition

$$\frac{\partial \psi}{\partial t} = 0 \qquad \forall \psi \in \{\text{Physical quantities}\} \tag{2.14}$$

An immediate important consequence of the stationary regime for Eq. (2.13) is

$$\frac{\partial}{\partial t} \rho_{\mathcal{V}} = 0 \Rightarrow \text{div} \vec{J} = 0 \tag{2.15}$$

the latter of which can be considered as the continuity equation in the stationary regime. Of course, $\frac{\partial}{\partial t} J_k = 0$ as well.

It is this regime on which the following theory of magnetostatics will be based on.

Chapter 3

Magnetostatics

3.1 Experimental Evidence

Magnetic phenomena have been known to mankind for thousands of years, because magnetised materials can be found on the earth's surface. The earliest scientific records of magnetism are known from ancient Greece. The modern scientific theory of magnetism dates back only a few hundred years, with milestones being the work of Romagnosi (1801, battery and magnetic needle), Ørsted (1819, electric currents create magnetic fields), Biot and Savart (1820, law of), and Ampère (1820-1825, law of).

The essential basic experiment measures forces between different arrangements of conducting wires carrying electric currents (Ørsted, Ampère). First, we define the direction of electric currents perpendicular to a plane of representation (here the piece of paper in front of you), see Figure 3.1. The major results of the experiments are summarized

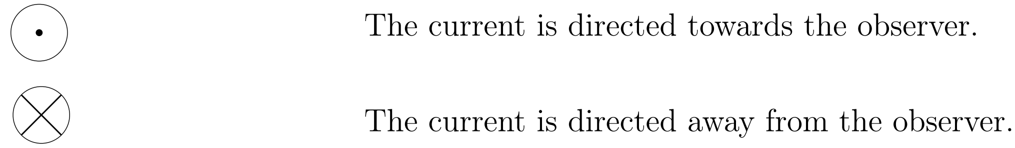
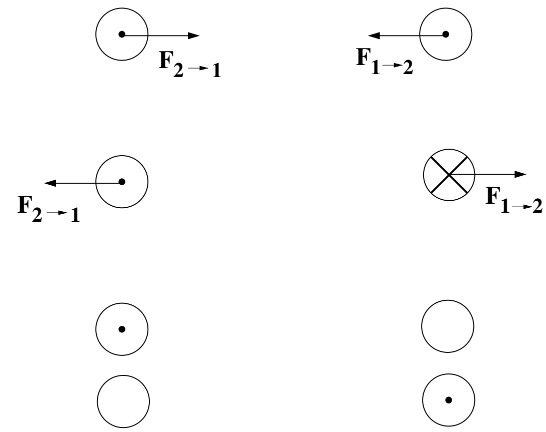


Figure 3.1: Definition of current direction

in Figure 3.2. From detailed studies with varying wire configurations and current intensities, a theory of the magnetic field could be con-



An attractive force acts on parallel wires with parallel currents.

A repulsive force acts on parallel wires with antiparallel currents.

No force at all is detected if one or both of the currents are interrupted.

Figure 3.2: Basic electromagnetic experiment; wire 1 is to the left, wire 2 to the right.

ceived, along the following qualitative lines: If it is assumed that the magnetic force $\vec{F}_m$ acting on the wires derives from a relation between an elementary current[1] $q\,\vec{v}$ and a magnetic vector field $\vec{B}$ *via* a vector product, as

$$\vec{F}_m = q\,\vec{v} \times \vec{B}, \tag{3.1}$$

then we can picture the various vector fields as done in Figures 3.3 and 3.4 for the uppermost configuration of wires in Fig. 3.2. $\vec{B}_2(\vec{r}_1)$ is to

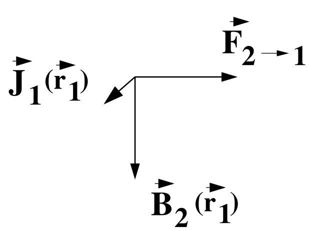


Figure 3.3: Force due to current in wire 1 and magnetic field vector at wire 1 due to current in wire 2

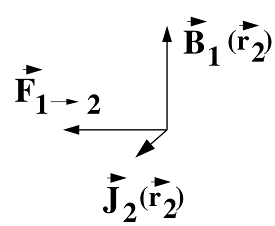


Figure 3.4: Force due to current in wire 2 and magnetic field vector at wire 2 due to current in wire 1

be understood as the magnetic field produced by the current in wire 2 (index "2") at the position of wire 1 (represented by the position vector $\vec{r}_1$). Here we use the right-hand convention as introduced in subsection 1.6.1.1. Since $\vec{B}$ is induced by moving charges it is also called magnetic

[1]The theory has been formulated for **stationary** currents, i.e. conductors carrying many elementary charges. The "current" $q\,\vec{v}$ should therefore be understood as an element of a stationary current in this context.

induction.

The magnetic force in Eq. 3.1 therefore results from the movement of charge q in the magnetic field $\vec{B}$ at the position of the charge[2].

3.2 The Lorentz Force

We are now in the position to present the first quantity that combines electro- and magnetostatics, which is of such fundamental character that it deserves a subsection of its own.

The total force acting on a charged particle is the sum of an electric and a magnetic force. With the electrostatic force in terms of the electric field from section 1.3.2 and Eq. (3.1) we obtain the **Lorentz force**[3]:

$$\vec{F}_L = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{3.2}$$

The fields are taken at the position of the charge q.

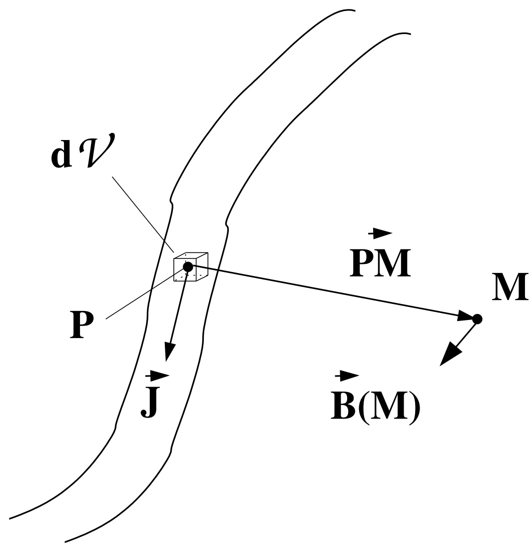
3.3 Biot and Savart's Law

After having established the principal form of the magnetic force acting on moving charges, we are still in need of a quantitative theory of the magnetic field due to a current, and the ensuing force on a charge.

Qualitatively, the direction of the magnetic field can again be conceived as resulting from a vector product, in this case of the electric current and a position vector to a specified point in space, as shown in Fig. 3.5. Quantitatively, the results of the aforementioned experiments have been summarized by Biot and Savart into the following elementary

[2] $\vec{F}_m$ is measured in this experiment as a force on the **wire**, the bulk mass of which is represented by the atomic nuclei of the metal the wire is made of. So the force on the conducting electrons is mediated onto the wire by the electron-proton interactions.

[3] The exact form of the Lorentz force has to account for kinematical effects due to Einstein's special theory of relativity, which introduces a prefactor. This will be discussed in the context of the relativistic representation of classical electrodynamics.



A magnetic field $\vec{B}$ in a point M due to a stationary volume current density $\vec{J}$

Figure 3.5: Qualitative representation of Biot and Savart's law

contribution to a magnetic vector field, due to an elementary stationary current $\vec{J}\, d\mathcal{V}$:

$$d\vec{B}(M) = \frac{\mu_0}{4\pi} \left(\vec{J}(P)\, d\mathcal{V}\right) \times \frac{\vec{PM}}{||\vec{PM}||^3} \tag{3.3}$$

The constant $\mu_0 = 4\pi \cdot 10^{-7}\,[J\, s^2\, C^{-2}\, m^{-1} = N\, A^{-2}]$ is the vacuum permeability. It has this exact value and is obtained through the measured force on parallel conducting wires and Ampère's force law

$$\frac{\Delta F_m}{\Delta L} = 2\frac{\mu_0}{4\pi}\,\frac{I_1\, I_2}{r} \tag{3.4}$$

where L is a length and r the distance between parallel wires. The S.I. unit of the magnetic field is Tesla, $T = N\, A^{-1}\, m^{-1}$.

Qualitatively, and using Biot-Savart's law, the magnetic field at the position of a wire in the uppermost configuration of Fig. 3.2 is as shown in Figs. 3.6 and 3.7: The magnetic fields are thus in accord with those assumed to be at the origin of the magnetic force introduced above.

We can regard Eq. (3.3) as the magnetism elementary equivalent of Eq. (1.15). This implies that where electric charges are the genera-

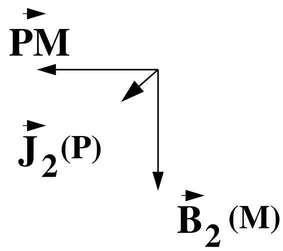


Figure 3.6: Magnetic field vector at wire 1 due to current in wire 2

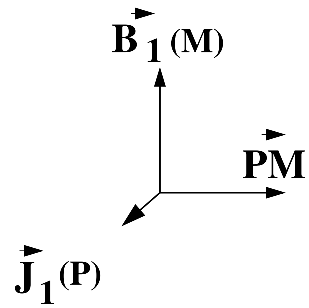


Figure 3.7: Magnetic field vector at wire 2 due to current in wire 1

tors of the electric field, **electric currents** are the generators of the **magnetic field**.

The total magnetic field in a point M is obtained by integrating the expression for the elementary contribution, Eq. (3.3), over all space:

$$\vec{B}(M) = \frac{\mu_0}{4\pi} \iiint_{\mathcal{V}} \vec{J}(P) \times \frac{\vec{PM}}{||\vec{PM}||^3} \, d\mathcal{V} \quad (3.5)$$

Very often we assume an idealized situation where the conducting medium is simplified to be an infinitely thin wire. In that case the volume integral over the current density can be written as

$$\iiint_{\mathcal{V}} \vec{J} \, d\mathcal{V} = \int_{\mathcal{C}} \left(\iint_{\mathcal{S}} \vec{J} \cdot \vec{n} \, dS \right) \vec{n} \, d\ell \longrightarrow \int_{\mathcal{C}} I \, \vec{d\ell} \quad (3.6)$$

because $\vec{J} || \vec{n}$ is here always fulfilled. Effectively, the volume integration has thus become a linear integration along the path of the conducting wire.

Consequently, Biot and Savart's law can be written for linear current densities as

$$\vec{B}(M) = \frac{\mu_0 \, I}{4\pi} \oint_{\mathcal{C}} \vec{dl}(P) \times \frac{\vec{PM}}{||\vec{PM}||^3} \quad (3.7)$$

3.4 Properties of the magnetic field

3.4.1 Symmetry of the Magnetostatic Field

As already introduced in section 1.5 for electrostatics, we will now establish the symmetry properties of the magnetic field based on the **symmetry of** the underlying (stationary) **currents**.

Suppose we have a current distribution with a plane of symmetry[4], as shown in Fig. 3.8. We immediately deduce, using the differential

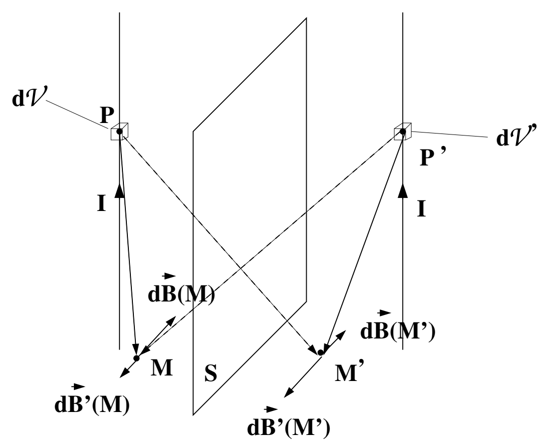
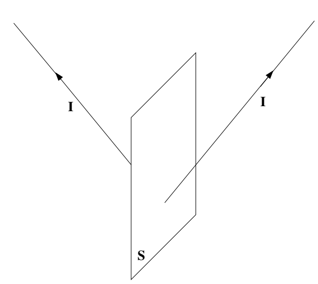


Figure 3.8: Stationary current distribution with a symmetry plane S. Two points M and M' are introduced at equal distance from S.

form of Biot-Savart's law, Eq. (3.3) for a linear current, the following

[4] In the present case only the tangential component of $\vec{I}$ with respect to the plane of symmetry exists. In a more general setting, we must also consider the normal component, e.g. for the here shown distribution. For a plane of symmetry of the current, the normal component changes sign upon reflection, $I_n \to -I_n$ whereas the tangential component retains the same sign, $I_t \to I_t$.



elementary field contributions:

$$\begin{aligned}
d\vec{B}(M) &= \frac{\mu_0\, I}{4\pi}\, \vec{dl} \times \frac{\vec{PM}}{||\vec{PM}||^3} \\
d\vec{B}'(M) &= \frac{\mu_0\, I}{4\pi}\, \vec{dl}' \times \frac{\vec{P'M}}{||\vec{P'M}||^3} \\
d\vec{B}(M') &= \frac{\mu_0\, I}{4\pi}\, \vec{dl} \times \frac{\vec{PM'}}{||\vec{PM'}||^3} \\
d\vec{B}'(M') &= \frac{\mu_0\, I}{4\pi}\, \vec{dl}' \times \frac{\vec{P'M'}}{||\vec{P'M'}||^3} \qquad (3.8)
\end{aligned}$$

Due to the symmetry plane relating the currents we have $\vec{dl} = \vec{dl}'$. Furthermore, the arrangement of points at symmetric displacements from the symmetry plane allows to write $||\vec{P'M'}|| = ||\vec{PM}||$ and $||\vec{PM'}|| = ||\vec{P'M}||$. Therefore,

$$\begin{aligned}
d\vec{B}(M) + d\vec{B}'(M) &= \frac{\mu_0\, I}{4\pi} \left[\vec{dl} \times \frac{\vec{PM}}{||\vec{PM}||^3} + \vec{dl} \times \frac{\vec{P'M}}{||\vec{P'M}||^3} \right] \\
d\vec{B}(M') + d\vec{B}'(M') &= \frac{\mu_0\, I}{4\pi} \left[\vec{dl} \times \frac{\vec{PM'}}{||\vec{P'M}||^3} + \vec{dl} \times \frac{\vec{P'M'}}{||\vec{PM}||^3} \right] (3.9)
\end{aligned}$$

Since the four points P, P', M, M' in Fig. 3.8 are located in a plane, we obtain the same relationships between normal and tangential components of the connecting vectors as in Eq. (1.44). Writing the total contribution to the field in point M as $d\vec{B}(M)$ and the total contribution in point M' as $d\vec{B}(M')$, we can make the following identifications:

$$\begin{aligned}
d\vec{B}(M)_t &= \frac{\mu_0\, I}{4\pi}\left[\vec{dl}\times\frac{\vec{PM}_n}{||\vec{PM}||^3}+\vec{dl}\times\frac{\vec{P'M}_n}{||\vec{P'M}||^3}\right]\\
d\vec{B}(M)_n &= \frac{\mu_0\, I}{4\pi}\left[\vec{dl}\times\frac{\vec{PM}_t}{||\vec{PM}||^3}+\vec{dl}\times\frac{\vec{P'M}_t}{||\vec{P'M}||^3}\right]\\
d\vec{B}(M')_t &= \frac{\mu_0\, I}{4\pi}\left[-\vec{dl}\times\frac{\vec{P'M}_n}{||\vec{P'M}||^3}-\vec{dl}\times\frac{\vec{PM}_n}{||\vec{PM}||^3}\right]\\
d\vec{B}(M')_n &= \frac{\mu_0\, I}{4\pi}\left[\vec{dl}\times\frac{\vec{P'M}_t}{||\vec{P'M}||^3}+\vec{dl}\times\frac{\vec{PM}_t}{||\vec{PM}||^3}\right] \qquad (3.10)
\end{aligned}$$

Note that due to the vector product the normal components of the connecting vectors, $\vec{PM}_n$ and $\vec{P'M}_n$, are related to the **tangential** components of the magnetic field, and vice versa. We finally obtain by comparing normal and tangential components in the preceding equation

$$\boxed{\begin{aligned}
\vec{B}_n(M) &= \vec{B}_n(M')\\
\vec{B}_t(M) &= -\vec{B}_t(M') \quad (3.11)
\end{aligned}}$$

which is just the exact opposite of the symmetry relations obtained for the electrostatic field, Eq. (1.46).

3.4.1.1 Points in Planes of Symmetry

For the special case of points in planes of symmetry, $M = M'$, and we obtain

$$\vec{B}_t(M) = -\vec{B}_t(M) = \vec{0}$$

leaving $\vec{B}_n(M)$ as the only non-vanishing component. In other words, if a current distribution displays a symmetry plane, then the magnetic field in points within this plane is perpendicular to the plane, $\vec{B} \perp \text{plane}(S)$.

3.4.1.2 Points in Planes of Antisymmetry

In this case one of the currents has been reversed, compared to the afore-discussed situation, so $\vec{dl} = -\vec{dl}'$. This situation is shown in Fig. 3.9. Then Eqs. (3.10) become

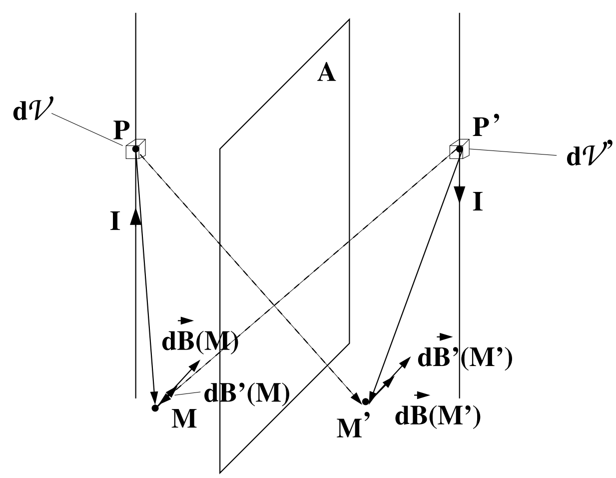


Figure 3.9: Stationary current distribution with an antisymmetry plane A. Two points M and M' are introduced at equal distance from A.

$$\begin{aligned}
d\vec{B}(M)_t &= \frac{\mu_0 I}{4\pi}\left[\vec{dl} \times \frac{\vec{PM}_n}{||\vec{PM}||^3} - \vec{dl} \times \frac{\vec{P'M}_n}{||\vec{P'M}||^3}\right] \\
d\vec{B}(M)_n &= \frac{\mu_0 I}{4\pi}\left[\vec{dl} \times \frac{\vec{PM}_t}{||\vec{PM}||^3} - \vec{dl} \times \frac{\vec{P'M}_t}{||\vec{P'M}||^3}\right] \\
d\vec{B}(M')_t &= \frac{\mu_0 I}{4\pi}\left[-\vec{dl} \times \frac{\vec{P'M}_n}{||\vec{P'M}||^3} + \vec{dl} \times \frac{\vec{PM}_n}{||\vec{PM}||^3}\right] \\
d\vec{B}(M')_n &= \frac{\mu_0 I}{4\pi}\left[\vec{dl} \times \frac{\vec{P'M}_t}{||\vec{P'M}||^3} - \vec{dl} \times \frac{\vec{PM}_t}{||\vec{PM}||^3}\right] \qquad (3.12)
\end{aligned}$$

and now the result of comparison is

$$\begin{aligned} \vec{B}_n(M) &= -\vec{B}_n(M') \\ \vec{B}_t(M) &= \vec{B}_t(M'). \end{aligned} \quad (3.13)$$

If the point in question is located in the antisymmetry plane, $M = M'$, then

$$\vec{B}_n(M) = -\vec{B}_n(M) = \vec{0}$$

and the magnetic field lies within the plane, $\vec{B} \,||\, \text{plane}(A)$.

We conclude that the reversion of the symmetry properties relative to the electrostatic field is essentially due to the vector product relationship between magnetic field and its source, the current. In contrast to this, the electrostatic field is oriented along the line connecting field point and source point.

3.4.2 Ampère's Theorem. Integral Form

In electrostatics, charges are the sources of the electrostatic field, and the integral relationship that arises is Gauss's Theorem. We will now study the relationship between the magnetic field and its sources (electric currents) in order to establish an equation which can be regarded as the **magnetism equivalent of Gauss's Theorem**.

Our goal is to derive an expression for the line integral over the magnetostatic field. Be a magnetic field due to a circular loop of current $\mathcal{C}$ as shown in Fig. 3.10. Since the magnetic field at point M is given by virtue of Biot-Savart's law, Eq. (3.7), we can use the distance vector $\vec{dr}$ connecting the points M and M' along a path of integration $\mathcal{C}'$ to define an elementary line integral element (circulation)

$$\vec{B} \cdot \vec{dr} = \frac{\mu_0 \, I}{4\pi} \left(\oint_{\mathcal{C}} \vec{dl}(P) \times \frac{\vec{PM}}{||\vec{PM}||^3} \right) \cdot \vec{dr}. \qquad (3.14)$$

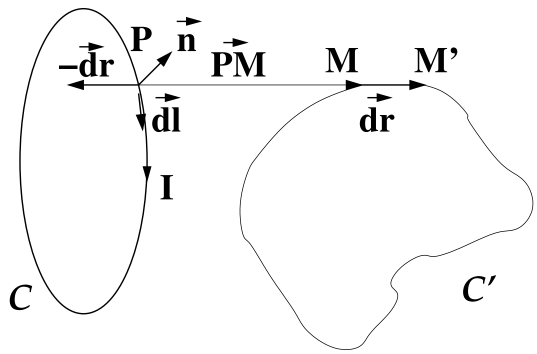


Figure 3.10: Basic configuration for the derivation of the line integral over the magnetic field.

With the help of some vector analysis, here the circular identities (valid for commuting vector components) $\left(\vec{a} \times \vec{b}\right) \cdot \vec{c} = \left(\vec{b} \times \vec{c}\right) \cdot \vec{a} = (\vec{c} \times \vec{a}) \cdot \vec{b}$, Eq. (3.14) can be rewritten as

$$\vec{B} \cdot \vec{dr} = \frac{\mu_0 I}{4\pi} \oint_{\mathcal{C}} \frac{\vec{MP}}{||\vec{PM}||^3} \cdot \left(-\vec{dr} \times \vec{dl}\right), \tag{3.15}$$

where the second of the circular identities and the commutation of the scalar product have been used. We now obtain the line integral by integrating over the entire path $\mathcal{C}'$:

$$\oint_{\mathcal{C}'} \vec{B} \cdot \vec{dr} = \frac{\mu_0 I}{4\pi} \oint_{\mathcal{C}'} \oint_{\mathcal{C}} \frac{\vec{MP}}{||\vec{PM}||^3} \cdot \left(-\vec{dr} \times \vec{dl}\right). \tag{3.16}$$

We can reinterpret the vector product under the integral by writing

$$-\vec{dr} \times \vec{dl} := \vec{n}\, dS \tag{3.17}$$

where the normal vector $\vec{n}$ is orthogonal to the surface element dS which arises from the orientations and lengths of $-\vec{dr}$ and $\vec{dl}$. This means that the right-hand side of Eq. (3.16) becomes a surface integration over an open surface δS which is created by the vector product following the integration paths:

$$\oint_{\mathcal{C}'} \vec{B} \cdot \vec{dr} = \frac{\mu_0 I}{4\pi} \oint_{\mathcal{C}'} \int_{\delta\mathcal{S}} \frac{\vec{MP}}{||\vec{PM}||^3} \cdot \vec{n}\, dS \tag{3.18}$$

δS can be understood as the surface that is created as the point P follows the integration path $\mathcal{C}$.

So the right-hand side is related to the flux of the vector field $\frac{\vec{MP}}{||\vec{PM}||^3}$ traversing this surface. However, due to Eq. (1.57) this flux also corresponds to a solid angle, which we call $d\Omega$. It corresponds to the solid angle under which δS is seen from the point M. Thus, the expression can also be written as an integral over a solid angle, in accord with Eq. (1.56), as

$$\oint_{\mathcal{C}'} \vec{B} \cdot \vec{dr} = \frac{\mu_0 I}{4\pi} \oint_{\mathcal{C}'} d\Omega = \frac{\mu_0 I}{4\pi} \Delta\Omega, \tag{3.19}$$

where $\Delta\Omega = \Omega_{\text{final}} - \Omega_{\text{initial}}$ is the difference between solid angles in a final and an initial point of the integration path. We have in the following to distinguish between two cases:

1. The integration path does not link the circuit. In this case Eq.

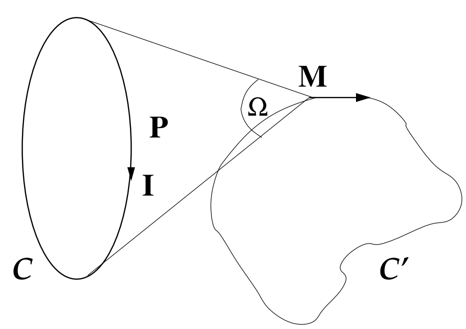


Figure 3.11: Solid angle for circuit and integration path that are not interlaced.

   (3.19) becomes

$$\oint_{\mathcal{C}'} \vec{B} \cdot \vec{dr} = \frac{\mu_0 I}{4\pi} \int_M^M d\Omega = \Omega(M) - \Omega(M) = 0. \tag{3.20}$$

2. The integration path links the circuit, as shown in Fig. 3.12. If the integration path along $\mathcal{C}'$ follows the orientation as in the figure,

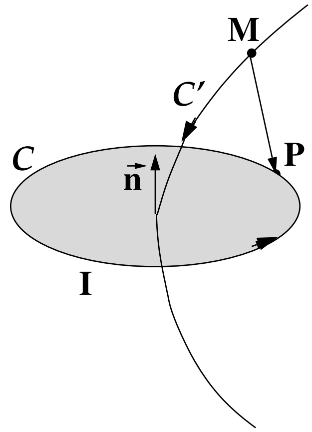


Figure 3.12: Solid angle discontinuity for circuit and integration path that are interlaced.

then $\Omega = -2\pi$ when M reaches the center of the loop $\mathcal{C}$. However, as M passes through the loop, there is a **discontinuity** of the solid angle, as it depends on the orientation of the surface (given by the orientation of $\mathcal{C}$). The solid angle changes from $\Omega_{\text{top}} = \pm 2\pi$ to $\Omega_{\text{bottom}} = \mp 2\pi$ when traversing the surface. So the integration picks up a factor of $\pm 4\pi$ in general. The sign of this factor can be determined in the following way: Due to Biot-Savart's law the mag-

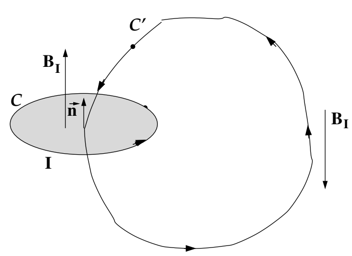


Figure 3.13: Integration path passing through surface opposed to surface orientation.

netic field $\vec{B}_I$ due to the current intensity I is oriented in the way as shown in Fig. 3.13. This means that the parallel component of the integration path element $\vec{dr}$ is always opposed (anti-collinear) with $\vec{B}_I$, and so the sign of $\vec{B} \cdot \vec{dr}$ is negative, in this case.

We can summarize the above analysis and state Ampère's theorem in integral form: If I_i is a linear current intensity in a circuit $\mathcal{C}_i$ that enlaces the integration contour $\mathcal{C}'$, then

$$\oint_{\mathcal{C}'} \vec{B} \cdot \vec{dr} = \mu_0 \sum_{i=1}^{n} s_i \, I_i \tag{3.21}$$

The sign s_i is determined by the orientation of I_i for every current loop that passes through the integration path, and by the orientation of the integration path itself.

The theorem is given a more general form using the definition of the linear current density Eq. (2.8),

$$\boxed{\oint_{\mathcal{C}} \vec{B} \cdot \vec{dr} = \mu_0 \iint_{\mathcal{S}} \vec{J} \cdot \vec{n} \, dS} \tag{3.22}$$

with $\vec{J}$ the volume current density traversing the oriented surface $\mathcal{S}$ which is delimited by the contour $\mathcal{C}$ (we have dropped the primes for simplicity).

3.4.3 Ampère's Theorem. Local Form

We start out by realizing that a closed integration path $\mathcal{C}$ can be represented by two closed paths $\mathcal{C}_1$ and $\mathcal{C}_2$ that are enclosed by the first, as shown in Fig. 3.14. A line integral over $\mathcal{C}$ can then be rewritten using the two new paths. In the case of the magnetostatic field this means

$$\oint_{\mathcal{C}} \vec{B} \cdot \vec{dr} = \oint_{\mathcal{C}_1} \vec{B} \cdot \vec{dr} + \oint_{\mathcal{C}_2} \vec{B} \cdot \vec{dr}, \tag{3.23}$$

since obviously

$$\int_{A}^{B} \vec{B} \cdot \vec{dr} = - \int_{B}^{A} \vec{B} \cdot \vec{dr}. \tag{3.24}$$

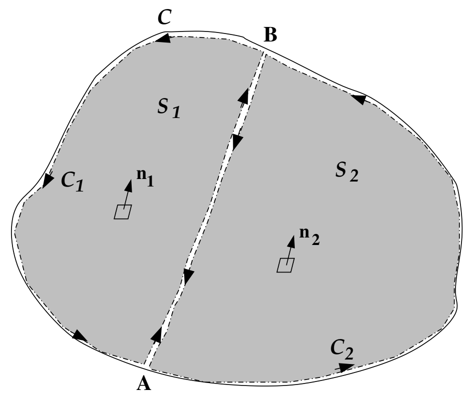


Figure 3.14: Subdivision of a closed integration path into two.

Under the assumption that the volume current density $\vec{J}$ traversing the contour in question is homogeneous (it is also stationary by definition in the current context), and using Ampère's theorem, Eq. (3.22), Eq. (3.23) becomes

$$\oint_{\mathcal{C}_1(\mathcal{S}_1)} \vec{B} \cdot d\vec{r} + \oint_{\mathcal{C}_2(\mathcal{S}_2)} \vec{B} \cdot d\vec{r} = \mu_0 \left[\iint_{\mathcal{S}_1} \vec{J} \cdot \vec{n}_1 \, dS + \iint_{\mathcal{S}_2} \vec{J} \cdot \vec{n}_2 \, dS \right]. \tag{3.25}$$

Note that this identity holds for the situation depicted in Fig. 3.14. We may now continue to divide $\mathcal{S}$ into an arbitrary number N of connected subsurfaces, so that Eq. (3.25) turns into

$$\sum_{k=1}^{N} \oint_{\mathcal{C}_k(\mathcal{S}_k)} \vec{B} \cdot d\vec{r} = \mu_0 \sum_{k=1}^{N} \iint_{\mathcal{S}_k} \vec{J} \cdot \vec{n}_k \, dS. \tag{3.26}$$

In a next step we consider a single of these loops and the surface it encloses, premultiply this equation with a factor $\frac{1}{\mathcal{S}_k}$ and take the infinitesimal limit, i.e., we let the fragment surface and therefore also the contour delimiting it tend to zero:

$$\lim_{\mathcal{S}_k \to 0} \frac{1}{\mathcal{S}_k} \oint_{\mathcal{C}_k} \vec{B} \cdot d\vec{r} = \mu_0 \lim_{\mathcal{S}_k \to 0} \frac{1}{\mathcal{S}_k} \iint_{\mathcal{S}_k} \vec{J} \cdot \vec{n}_k \, dS \tag{3.27}$$

The left-hand side of this equation gives, multiplied by the normal vector onto the respective $\mathcal{S}_k$, the **definition of the curl of a vector field**, here the vector field $\vec{B}$:

$$\vec{\text{rot}}\vec{B} \cdot \vec{n}_k := \lim_{\mathcal{S}_k \to 0} \frac{1}{\mathcal{S}_k} \oint_{\mathcal{C}_k(\mathcal{S}_k)} \vec{B} \cdot d\vec{r} \qquad (3.28)$$

In the limit $\mathcal{S}_k \to 0$ the integral on the right-hand side of Eq. (3.27) becomes equal to $\vec{J} \cdot \vec{n}_k \, \mathcal{S}_k$, and so it can be inferred that at the point P around which $\mathcal{S}_k$ is taken

$$\vec{\text{rot}}\vec{B}(P) \cdot \vec{n}_k = \mu_0 \, \vec{J}(P) \cdot \vec{n}_k \qquad (3.29)$$

Therefore,

$$\vec{\text{rot}}\vec{B} = \mu_0 \, \vec{J} \qquad (3.30)$$

which constitutes Ampère's Theorem in local form. It means that the curl of the magnetic vector field in a point is non-zero only if the volume current density in that point is non-zero. Restated in other words, in a region without sources (currents) the curl of the magnetostatic field vanishes.

3.4.4 Calculus for the Curl

Using arguments similar to those already developed for the case of the divergence operation in subsection 1.7.1 for Eq. (3.28), we can write the curl operation on a vector field $\vec{G}$ in cartesian coordinates as

$$\vec{\text{rot}}\vec{G} = \boldsymbol{\nabla} \times \vec{G}. \qquad (3.31)$$

Using the Lévy-Civita symbol we can also write

$$\left(\sum_{i=1}^{3} \vec{e}_i \, \frac{\partial}{\partial x_i} \right) \times \left(\sum_{i=1}^{3} \vec{e}_j \, G_j(\vec{x}) \right) = \sum_{i,j,k=1}^{3} \varepsilon_{ijk} \, \frac{\partial}{\partial x_i} \, G_j(\vec{x}) \, \vec{e}_k. \qquad (3.32)$$

3.4.5 Stokes' Theorem

If we take the flux through a surface $\mathcal{S}$ on both sides of Ampère's Theorem, Eq. (3.30), we arrive at

$$\iint\limits_{\mathcal{S}} \vec{\text{rot}}\vec{B} \cdot \vec{n}\, dS = \mu_0 \iint\limits_{\mathcal{S}} \vec{J} \cdot \vec{n}\, dS, \tag{3.33}$$

and using Ampère's Theorem in integral form, Eq. (3.22) for the right-hand side, Eq. (3.33) becomes

$$\iint\limits_{\mathcal{S}} \vec{\text{rot}}\vec{B} \cdot \vec{n}\, dS = \oint\limits_{\mathcal{C}(\mathcal{S})} \vec{B} \cdot \vec{dr} \tag{3.34}$$

which is known as **Stokes' Theorem** or curl theorem. In the present case it has been derived for the magnetostatic field, but it may, just like the theorem of Gauss and Ostrogradsky, be generalized to arbitrary vector fields $\vec{G}$, i.e.,

$$\boxed{\iint\limits_{\mathcal{S}} \vec{\text{rot}}\, \vec{G} \cdot \vec{n}\, dS = \oint\limits_{\mathcal{C}(\mathcal{S})} \vec{G} \cdot \vec{dr} \quad (3.35)}$$

3.5 Conservation of the Flux of the Magnetostatic Field

Stokes' Theorem, Eq. (3.34) tells us something about the flux of the *curl* of a vector field through a delimited surface. But what about the flux of the vector field itself? We know from subsection 1.7.4 and the discussion preceding that, that the flux of a vector field is related to its divergence. So let us first determine the divergence of the magnetostatic field.

In accord with Fig. 3.15, and using Biot-Savart's law in the form for a stationary current density, Eq. (3.5), the magnetostatic field in a

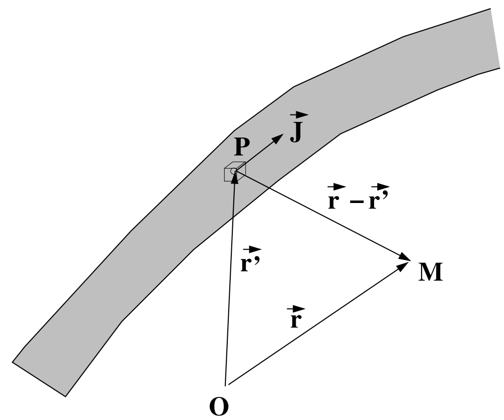


Figure 3.15: Stationary current density $\vec{J}$ and coordinates in a laboratory reference frame.

point M is given by using the coordinate vector $\vec{r}$ as

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \iiint\limits_{\mathcal{V}'} \vec{J}(\vec{r}\,') \times \frac{\vec{r} - \vec{r}\,'}{||\vec{r} - \vec{r}\,'||^3}\, d\mathcal{V}'. \tag{3.36}$$

We now calculate the divergence of this magnetostatic field:

$$\begin{aligned} \text{div}\vec{B}(\vec{r}) &= \boldsymbol{\nabla}_{\vec{r}} \cdot \vec{B}(\vec{r}) \\ &= \frac{\mu_0}{4\pi} \iiint\limits_{\mathcal{V}'} \boldsymbol{\nabla}_{\vec{r}} \cdot \left[\vec{J}(\vec{r}\,') \times \frac{\vec{r} - \vec{r}\,'}{||\vec{r} - \vec{r}\,'||^3}\, d^3r'\right] \\ &= \frac{\mu_0}{4\pi} \iiint\limits_{\mathcal{V}'} \boldsymbol{\nabla}_{\vec{r}} \cdot \left[\vec{J}(\vec{r}\,') \times \left(\boldsymbol{\nabla}_{\vec{r}} \frac{-1}{||\vec{r} - \vec{r}\,'||}\right)\right] d^3r' \end{aligned} \tag{3.37}$$

In the last line we have used the expression for the gradient of the scalar field $\frac{-1}{||\vec{r}-\vec{r}\,'||}$.

In the next step, we exploit the vector analysis relation (see Eq. (8) in appendix .14

$$\text{div}\left[\vec{a}(\vec{r}) \times \vec{b}(\vec{r})\right] = \vec{b} \cdot \vec{\text{rot}}\, \vec{a} - \vec{a} \cdot \vec{\text{rot}}\, \vec{b} \tag{3.38}$$

With the help of this relation Eq. (3.37) can be reformulated to become

$$\mathrm{div}\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \iiint\limits_{\mathcal{V}'} \left[\frac{\vec{r}-\vec{r}\,'}{||\vec{r}-\vec{r}\,'||^3} \cdot \vec{\mathrm{rot}}\, \vec{J}(\vec{r}\,') + \vec{J}(\vec{r}\,') \cdot \vec{\mathrm{rot}} \left(\vec{\mathrm{grad}}\, \frac{1}{||\vec{r}-\vec{r}\,'||} \right) \right] \tag{3.39}$$

Using the above-developed techniques for vector analysis we can demonstrate that

$$\vec{\mathrm{rot}} \left(\vec{\mathrm{grad}}\, f(\vec{r}) \right) = 0 \tag{3.40}$$

$$\vec{\mathrm{rot}}\, \vec{J}(\vec{r}\,') = 0 \tag{3.41}$$

and so it follows that there is another local relation for the magnetostatic field

$$\mathrm{div}\vec{B} = 0. \tag{3.42}$$

The fact that the magnetostatic field is divergence free is true independent of the presence of currents. It is therefore an intrinsic property of $\vec{B}$. In general, vector fields $\vec{G}$ that are divergence free in all points, $\mathrm{div}\,\vec{G} = 0$, are called **solenoidal fields**.

Since Gauss and Ostrogradsky's Theorem, Eq. (1.86), is valid for any vector field, we can write

$$\oiint\limits_{S(\mathcal{V})} \vec{B} \cdot \vec{n}\, dS = \iiint\limits_{\mathcal{V}} \mathrm{div}\vec{B}\, d\mathcal{V} \tag{3.43}$$

and using Eq. (3.42) we obtain

$$\oiint\limits_{S(\mathcal{V})} \vec{B} \cdot \vec{n}\, dS = 0. \tag{3.44}$$

Eqs. (3.42) and (3.44) are the **structure equations of the magnetostatic field in local and integral form**, respectively. They are one of Maxwell's Equations, in local and in integral form, respectively.

If we now construct two surfaces which are both delimited by a contour $\mathcal{C}$, one lying inside the other, then the respective fluxes of $\vec{B}$

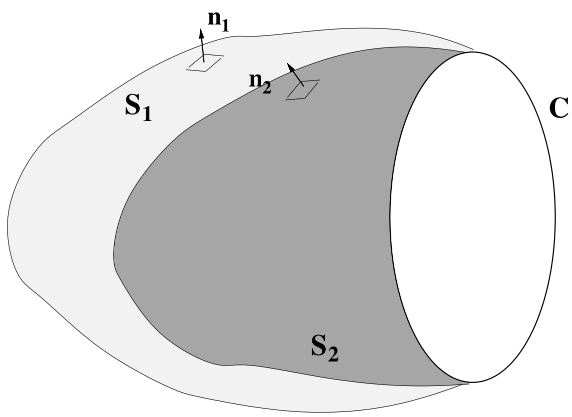


Figure 3.16: Two open surfaces delimited by the same contour C. S_2 comes to lie inside S_1.

traversing $\mathcal{S}_1$ and $\mathcal{S}_2$ are

$$\Phi_1 = \iint_{\mathcal{S}_1} \vec{B} \cdot \vec{n}_1 \, dS$$

$$\Phi_2 = \iint_{\mathcal{S}_2} \vec{B} \cdot \vec{n}_2 \, dS$$

If the union of the open surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ is considered, then we can write

$$\oiint_{\mathcal{S}_1 \cup \mathcal{S}_2} \vec{B} \cdot \vec{n} \, dS = \iint_{\mathcal{S}_2} \vec{B} \cdot \vec{n}_2 \, dS - \iint_{\mathcal{S}_1} \vec{B} \cdot \vec{n}_1 \, dS. \tag{3.45}$$

The sign change results from the fact that one of the surfaces is oriented towards the interior of the resulting closed surface. To convince yourself that this is true, take a look at Fig. (1.19) and the following equation.

Due to Eq. (3.44)

$$\iint_{\mathcal{S}_2} \vec{B} \cdot \vec{n}_2 \, dS - \iint_{\mathcal{S}_1} \vec{B} \cdot \vec{n}_1 \, dS = 0 \tag{3.46}$$

and it follows that

$$\boxed{\Phi_1 = \Phi_2, \qquad (3.47)}$$

independent of the specific choice of the two surfaces. This means that any surface chosen which is delimited by the respective contour produces the same flux. Thus, the flux of the magnetostatic field depends only on that contour, and it is therefore understood to have conservative flux.

3.6 Magnetic Vector Potential

3.6.1 Definition

In electrostatics the field derives from a scalar potential via a local relation, Eq. (1.36). We will next establish a corresponding potential in magnetostatics. The starting point is the central intrinsic property of the magnetostatic field, the fact that is it divergence free (Eq. (3.42))

$$\mathrm{div}\vec{B} = 0.$$

Let us take the divergence of the curl of an arbitrary vector field $\vec{G}(\vec{r})$. The result is

$$\mathrm{div}\left(\vec{\mathrm{rot}}\,\vec{G}(\vec{r})\right) = 0.$$

Since, the divergence of the curl of an arbitrary vector field vanishes, we can use this result and relate it to the magnetic field:

$$\boxed{\vec{B} = \vec{\mathrm{rot}}\,\vec{A} \qquad (3.48)}$$

It means that any arbitrary magnetic field can be written as the curl of an associated potential which itself is a vector field. $\vec{A}(\vec{r})$ is called the **vector potential** of the magnetic field.

3.6.2 Line integral over the vector potential

We write the flux of $\vec{B}$ traversing a surface delimited by a contour $\mathcal{C}$:

$$\begin{aligned}\Phi &= \iint\limits_{\mathcal{S}} \vec{B}\cdot\vec{n}\,dS = \iint\limits_{\mathcal{S}} \vec{\text{rot}}\,\vec{A}\cdot\vec{n}\,dS \\ &= \oint\limits_{\mathcal{C}(\mathcal{S})} \vec{A}\cdot\vec{dr} \end{aligned} \tag{3.49}$$

where in the last identity we have used Stokes' Theorem, Eq. (3.34), for the vector field $\vec{A}$. One thus arrives at a lemma: The flux of the magnetic field through a surface can also be calculated as the line integral over the corresponding vector potential, from which the field derives, for a contour delimiting the surface.

3.6.3 General Expression for the Vector Potential

Although the vector potential is an analogy to the scalar electrostatic potential, it is more difficult to handle than the latter and cannot always be given in a simple closed form. However, we can derive a general expression for $\vec{A}(\vec{r})$.

We start from Eq. (3.7), rewritten using the reference frame and position vectors as shown in Fig. 3.15. Then

$$\vec{B}(\vec{r}) = \frac{\mu_0\,I'}{4\pi}\oint\limits_{\mathcal{C}'} \vec{dr}\,' \times \frac{\vec{r}-\vec{r}\,'}{||\vec{r}-\vec{r}\,'||^3} \tag{3.50}$$

We reformulate the integrand as follows:

$$
\begin{aligned}
\vec{dr}' \times \frac{\vec{r}-\vec{r}'}{||\vec{r}-\vec{r}'||^3} &= -\vec{dr}' \times \boldsymbol{\nabla}_r \frac{1}{||\vec{r}-\vec{r}'||} \\
&= \boldsymbol{\nabla}_r \frac{1}{||\vec{r}-\vec{r}'||} \times \vec{dr}' + \frac{1}{||\vec{r}-\vec{r}'||}\, \boldsymbol{\nabla}_r \times \vec{dr}' \\
&= \boldsymbol{\nabla}_r \times \frac{\vec{dr}'}{||\vec{r}-\vec{r}'||} \qquad (3.51)
\end{aligned}
$$

where in the second line a zero has been added. Here it has, moreover, been used that in general

$$
\vec{\mathrm{rot}}\left(u(\vec{r})\,\vec{G}(\vec{r})\right) = \left(\vec{\mathrm{grad}}\, u(\vec{r})\right) \times \vec{G}(\vec{r}) + u(\vec{r})\,\vec{\mathrm{rot}}\,\vec{G}(\vec{r}) \qquad (3.52)
$$

and in our particular case $u(\vec{r}) := \frac{1}{||\vec{r}-\vec{r}'||}$ and $\vec{G}(\vec{r}) := \vec{dr}'$. With this reformulation the magnetic field from Eq. (3.50) becomes

$$
\begin{aligned}
\vec{B}(\vec{r}) &= \frac{\mu_0\, I}{4\pi} \oint_{\mathcal{C}'} \boldsymbol{\nabla}_r \times \frac{\vec{dr}'}{||\vec{r}-\vec{r}'||} \\
&= \boldsymbol{\nabla}_r \times \left(\frac{\mu_0\, I}{4\pi} \oint_{\mathcal{C}'} \frac{\vec{dr}'}{||\vec{r}-\vec{r}'||} \right) \qquad (3.53)
\end{aligned}
$$

and by comparison with Eq. (3.48) it immediately follows that

$$
\vec{A}(\vec{r}) = \frac{\mu_0\, I}{4\pi} \oint_{\mathcal{C}'} \frac{\vec{dr}'}{||\vec{r}-\vec{r}'||} \qquad (3.54)
$$

which is a general expression for the magnetic vector potential in case of a filamentary current density producing the magnetic field. This expression may be readily generalized to the case of a volume current density:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \iiint\limits_{\mathcal{V}'} \frac{\vec{J}(\vec{r}\,')}{||\vec{r} - \vec{r}\,'||}\, d\mathcal{V}' \qquad (3.55)$$

In the case of motional point charges, the charged current density from Eq. (2.6) will be used and the vector potential is then written as

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \sum_{j=1}^{n} q_j\, \vec{v}_j \iiint\limits_{\mathcal{V}'} \frac{\delta^{(3)}(\vec{r}\,' - \vec{r}_j)}{||\vec{r} - \vec{r}\,'||}\, d\mathcal{V}' \qquad (3.56)$$

Let us finally investigate the divergence of the magnetic vector potential. Starting from Eq. (3.54) for the case of a filamentary current

$$\begin{aligned} \boldsymbol{\nabla}_r \cdot \vec{A}(\vec{r}) &= \frac{\mu_0\, I}{4\pi} \oint\limits_{\mathcal{C}'} \boldsymbol{\nabla}_r \cdot \frac{d\vec{r}\,'}{||\vec{r} - \vec{r}\,'||} \\ &= -\frac{\mu_0\, I}{4\pi} \oint\limits_{\mathcal{C}'} d\vec{r}\,' \cdot \boldsymbol{\nabla}_{r'} \frac{1}{||\vec{r} - \vec{r}\,'||} \end{aligned} \qquad (3.57)$$

This holds true due to the general relation

$$\mathrm{div}\left(u(\vec{x})\, \vec{G}(\vec{x})\right) = u(\vec{x})\, \mathrm{div}\, \vec{G}(\vec{x}) + \vec{G}(\vec{x}) \cdot \vec{\mathrm{grad}}\, u(\vec{x})$$

and the fact that $\boldsymbol{\nabla}_r \frac{1}{||\vec{r}-\vec{r}\,'||} = -\boldsymbol{\nabla}_{r'} \frac{1}{||\vec{r}-\vec{r}\,'||}$. Eq. (3.57) is reformulated using Stokes' Theorem, Eq. (3.34) for the vector field $\boldsymbol{\nabla}_{r'} \frac{1}{||\vec{r}-\vec{r}\,'||}$ to

$$\boldsymbol{\nabla}_r \cdot \vec{A}(\vec{r}) = -\frac{\mu_0\, I}{4\pi} \iint\limits_{\mathcal{S}'} \left[\boldsymbol{\nabla}_{r'} \times \left(\boldsymbol{\nabla}_{r'} \frac{1}{||\vec{r} - \vec{r}\,'||}\right)\right] \cdot \vec{n}\, dS \qquad (3.58)$$

But due to the identity Eq. (3.40) the curl of a gradient always vanishes, and so the integrand becomes zero. Therefore,

$$\mathrm{div}\vec{A} = 0 \qquad (3.59)$$

for a vector potential produced by a filamentary current density. This result can be generalized to arbitrary stationary current densities and their associated vector potentials by starting from Eq. (3.55) and using the divergence theorem and a boundary surface of a very large volume on which the current density is zero.

3.7 Gauge Invariance and Electromagnetic Gauge

The electrostatic scalar potential $V(\vec{r})$ is determined up to a constant. This constant vanishes when the electrostatic field is calculated from the underlying potential, and so $\vec{E}(\vec{r})$ is invariant with respect to an infinite set of different potentials, given the limitation to an additive constant.

There is an analogy in magnetism. Here it results from the fact that $\vec{B} = \vec{\mathrm{rot}}\vec{A}$. Suppose we vary a vector potential by adding the gradient of an arbitrary scalar field $f(\vec{r})$:

$$\vec{A} \longrightarrow \vec{A}' = \vec{A} + \vec{\mathrm{grad}} f \tag{3.60}$$

Eq. (3.60) is called an electromagnetic **gauge transformation**. Now we calculate the magnetic field for the transformed vector potential, as

$$\begin{aligned} \vec{B}' &= \vec{\mathrm{rot}}\vec{A}' = \vec{\mathrm{rot}}\vec{A} + \vec{\mathrm{rot}}\left(\vec{\mathrm{grad}} f\right) \\ &= \vec{B}, \end{aligned} \tag{3.61}$$

which is true due to the linearity of the curl operation and due to the fact that the curl of any gradient vanishes, Eq. (3.40).

In turn, this means that the scalar field $f(\vec{r})$ can be chosen freely **without changing** the resulting magnetic field, and therefore without changing the physics of magnetism.

An important **choice** is the so-called **Coulomb gauge**, defined by the condition

$$\mathrm{div}\vec{A}' = 0. \tag{3.62}$$

Using the general gauge transformation, Eq. (3.60), we arrive at

$$\mathrm{div}\vec{A}' = \mathrm{div}\vec{A} + \mathrm{div}\left(\vec{\mathrm{grad}} f\right) = \Delta f, \tag{3.63}$$

where Eq. (3.59) has been used, and so

$$\Delta f(\vec{r}) = 0. \tag{3.64}$$

This means that in Coulomb gauge the scalar field is not just some scalar field, but one that satisfies Laplace's equation, Eq. (1.92). Physically, since Laplace's equation is valid in regions without electric sources, this gauge is also called **radiation gauge**[5].

Another gauge fixing often used in physics is the Feynman gauge, which will be discussed in more advanced courses of electromagnetism, in particular when the concepts of special Einsteinian relativity have been introduced.

3.8 Magnetic Multipole Expansion

We have already seen in section 1.9 how the electrostatic potential can be expanded into multipole terms. A corresponding expansion can be carried out for the magnetic vector potential $\vec{A}$.

The basis for the expansion is now a spatial region with an arbitrary **current density** distribution, $\vec{J}(\vec{r}\,')$, see Fig. (3.17). We start out from the general expression for the vector potential, Eq. (3.55),

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \iiint\limits_{\mathcal{V}'} \frac{\vec{J}(\vec{r}\,')}{||\vec{r} - \vec{r}\,'||} \, d\mathcal{V}'$$

and use Eq. (1.103) from the multipole expansion for the electrostatic

[5]The theory of electromagnetic radiation requires the departure from the stationary regime and the introduction of the time-dependent domain.

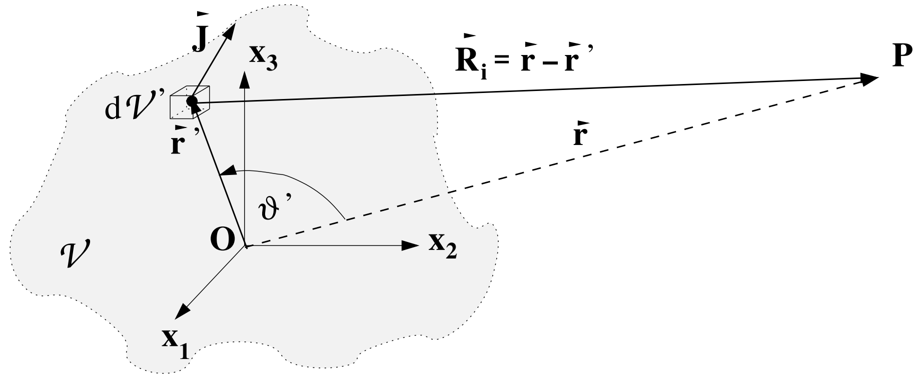


Figure 3.17: Current density distribution seen from a point P which is farther from the origin than any of the currents inside the respective region.

potential, where it was found that

$$\frac{1}{||\vec{r} - \vec{r}\,'||} = \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}}\, r'^{\ell}\, P_\ell(\cos\vartheta') \tag{3.65}$$

Combining the above two equations gives the vector potential as

$$\begin{aligned}
\vec{A}(\vec{r}) &= \frac{\mu_0}{4\pi} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \iiint\limits_{\mathcal{V}'} \vec{J}(\vec{r}\,')\, r'^{\ell}\, P_\ell(\cos\vartheta') d\mathcal{V}' \\
&= \frac{\mu_0}{4\pi r} \iiint\limits_{\mathcal{V}'} \vec{J}(\vec{r}\,')\, d\mathcal{V}' \qquad \text{monopole term}, \ell = 0 \\
&\quad + \frac{\mu_0}{4\pi r^2} \iiint\limits_{\mathcal{V}'} \vec{J}(\vec{r}\,')\, (\vec{e}_r \cdot \vec{r}\,')\, d\mathcal{V}' \qquad \text{dipole term}, \ell = 1 \\
&\quad + \frac{\mu_0}{4\pi r^3} \iiint\limits_{\mathcal{V}'} \frac{1}{2} \vec{J}(\vec{r}\,') \left[3\,(\vec{e}_r \cdot \vec{r}\,')^2 - r'^2 \right] d\mathcal{V}' \quad \text{quadrupole term}, \ell = 2
\end{aligned}$$

Just like for the electrostatic multipole expansion, the goal is to write the vector potential $\vec{A}(\vec{r})$ as a series in terms of multipole moments which are only functions of the current density distribution. The first term is the

3.8.1 Magnetic Monopole Moment

We begin with a manipulation of a component of the stationary current density distribution

$$J_x(\vec{r}\,') = \vec{J}(\vec{r}\,') \cdot \vec{e}_x + x' \vec{\nabla}_{\vec{r}\,'} \cdot \vec{J}(\vec{r}\,') \tag{3.66}$$

Due to the stationary form of the continuity equation (2.15) the term that has been added in Eq. (3.66) is zero, and so

$$J_x(\vec{r}\,') = \vec{\nabla}_{\vec{r}\,'} \cdot (x' \, J_x(\vec{r}\,'))\,. \tag{3.67}$$

This identity is seen to be true using relation (4).

The volume integral over $J_x(\vec{r}\,')$ is, therefore,

$$\iiint\limits_{\mathcal{V}'} J_x(\vec{r}\,')\, d\mathcal{V}' = \iiint\limits_{\mathcal{V}'} \vec{\nabla}_{\vec{r}\,'} \cdot \left(x' \, \vec{J}(\vec{r}\,') \right) \, d\mathcal{V}' = \oint\!\!\oint\limits_{\mathcal{S}'(\mathcal{V}')} x' \, \vec{J}(\vec{r}\,') \cdot \vec{n} \, d\mathcal{S}' \tag{3.68}$$

where the divergence theorem Eq. (1.86) has been used. But since the current density distribution is, by definition, localized in space, the boundary surface $\mathcal{S}(\mathcal{V})$ can always be chosen such that there is no flux of $\vec{J}(\vec{r}\,')$ traversing it. Thus, $\iiint_{\mathcal{V}'} J_i(\vec{r}\,') d\mathcal{V}' = 0$, $\forall\, i \in \{1, 2, 3\}$.

This means that there are **no magnetic monopoles in the framework of magnetostatics**[6].

3.8.2 Magnetic Dipole Moment

[TO BE CONTINUED ... ACTUALLY, THIS IS WHERE THE REAL FUN BEGINS.]

3.9 Poisson's Equation in Magnetism

Poisson's equation in electrostatics, Eq. (1.91), relates the electrostatic potential and its sources, electric charges. We wish to obtain a corre-

[6] The analogy in electrostatics is $\iiint_{\mathcal{V}'} \rho(\vec{r}\,')\, d\mathcal{V}' = Q_{\text{in}}$ pointing to the existence of electric monopoles.

sponding local relation between the vector potential and the sources of the magnetic field, electric currents.

The magnetic field is divergence free, so it is of interest to study its curl in the light of the relationship between magnetic field and vector potential. Starting from Ampère's Theorem in local form, Eq. (3.30), we have

$$\vec{\mathrm{rot}}\vec{B} = \vec{\mathrm{rot}}\left(\vec{\mathrm{rot}}\vec{A}\right) = \mu_0\,\vec{J} \tag{3.69}$$

It can be shown that a double curl of a vector field can be written as

$$\vec{\mathrm{rot}}\left(\vec{\mathrm{rot}}\vec{A}\right) = \vec{\mathrm{grad}}\left(\mathrm{div}\vec{A}\right) - \Delta\,\vec{A}, \tag{3.70}$$

and so, using Coulomb gauge, Eq. (3.63),

$$\Delta\,\vec{A} = -\mu_0\,\vec{J} \tag{3.71}$$

which is Poisson's equation in magnetism.

As a lemma, we here add the corresponding relation for electrostatics. Taking the curl of the electric field yields

$$\vec{\mathrm{rot}}\vec{E} = -\vec{\mathrm{rot}}\left(\vec{\mathrm{grad}}V\right) = 0, \tag{3.72}$$

due to Eq. (3.40). Thus, the electric field is curl free which consitutes the **local form of one Maxwell's Equations** already encountered in electrostatics, Eq. (1.27),

$$\boxed{\vec{\mathrm{rot}}\vec{E} = 0.} \tag{3.73}$$

Chapter 4

Summary: Maxwell's Equations in the Stationary Regime

Local form

$\vec{\text{rot}}\vec{E} = \vec{0}$	$\text{div}\vec{B} = 0$	Structure equations
$\text{div}\vec{E} = \frac{\rho}{\varepsilon_0}$	$\vec{\text{rot}}\vec{B} = \mu_0\,\vec{J}$	Relation between field and source

Integral form

$\oint\limits_{\mathcal{C}} \vec{E} \cdot \vec{dr} = 0$	$\oiint\limits_{\mathcal{S}} \vec{B} \cdot \vec{n}\, dS = 0$
$\oiint\limits_{\mathcal{S}} \vec{E} \cdot \vec{n}\, dS = \frac{Q}{\varepsilon_0}$	$\oint\limits_{\mathcal{C}} \vec{B} \cdot \vec{dr} = \mu_0\, I_t$

.1 Exercises 1: Electrostatic field, discrete distributions

1. **Coulomb's law, electrostatic field, electrostatic force**
   Be a two-dimensional cartesian coordinate system and a point charge $q_0 = -e$ located at the point $(0, 3)$, $q_1 = +2e$ at $(-3, -1)$, and $q_2 = -q_1$ at $(3, -1)$.
   (a) Calculate the total electrostatic field $\vec{E}(M)$ at the location of q_0 (M) due to the two other charges q_1 and q_2.
   (b) Determine the total electrostatic force acting on q_0.
   (c) Show that sum of the norms of the two individual forces does not equal the norm of the total force vector.
   (d) Calculate the norm of the total force acting on q_0 in units of the $S.I.$.
   (e) Visualize the different contributions to $\vec{E}(M)$ and $\vec{F}(M)$ graphically.
2. **Linear charge distribution**
   Be a semi-circle of radius R carrying a homogeneous linear charge density λ. Calculate the electrostatic field at the center of the corresponding circle as a function of λ.

Homework

1. *Be a hemisphere of radius R carrying a homogeneous surface charge density σ. Calculate the electrostatic field at the center of the corresponding sphere as a function of σ.*

2. *Consider two point charges $q_1 = +e$ and $q_2 = -e$ at a distance d. Determine a few field lines of the resulting dipole field by determining the direction of the electrostatic field at a number of chosen points. Trace the resulting field lines.*

.2 Exercises 2: Electrostatic field and potential, continuous distributions

1. We consider an idealized circular disc centered at O and perpendicular to the axis $z'Oz$, of outer radius $r = b$ and inner radius $r' = a$. The disc is uniformly charged, and its surface charge density be σ.
   (a) Determine the electrostatic field $\vec{E}$ and the electrostatic potential V at an arbitrary point M along the z axis. We set $V(\infty) = 0$.
   (b) Use the results of the preceding exercise to deduce:
      i. The electrostatic field and potential of a full disc of radius R at an arbitrary point along the z axis,
      ii. the electrostatic field created by a plane of infinite size.

Homework:

Be a positive point charge q at the origin of a cartesian coordinate frame.

1. *We consider two concentric circles C_1 and C_2 around the origin of radii r_1 and r_2.*
   (a) Calculate the line integral of the electrostatic field created by q for the path between two points $A_0(r_1, \varphi_0)$ and $A_1(r_1, \varphi_1)$ along C_1.
   (b) Calculate the line integral of the electrostatic field for the path from $A_0(r_1, \varphi_0)$ to $B_0(r_2, \varphi_0)$, which is located on C_2.
   (c) Calculate the line integral for two different paths from $A_0(r_1, \varphi_0)$ to $B_1(r_2, \varphi_1)$.
2. *Calculate the electrostatic field of a point charge using the local relation between electrostatic potential and electrostatic field and the "nabla" operator.*

.3 Exercises 3: Symmetry planes and the electrostatic field

1. Determine for the following cases all possible unique planes of symmetry and/or antisymmetry, and give the direction of the electrostatic field in every point lying in such a plane,

   (a) for a muon[1] μ,

   (b) an electric dipole, consisting of an electron e^- and a positron e^+ at a finite distance,

   (c) an electric quadrupole, formed by placing a π meson (π^+) at position $(a, 0, 0)$ and another at $(-a, 0, 0)$, respectively, and a π^- at position $(0, a, 0)$ and another at $(0, -a, 0)$, respectively, of a cartesian coordinate frame.
   Determine the direction of the electrostatic field at points $(0, 0, z)$, $(x, 0, 0)$, $(0, y, 0)$, $(x, x, 0)$, $(x, y, 0)$.

Homework:

Determine all possible unique planes of symmetry and/or antisymmetry for a uniformly and positively charged finite cylinder. Determine the direction of the electrostatic field for all kinds of points.

[1] A muon is a lepton of the second generation, carrying a charge $-e$.

.4 Exercises 4: Gauss's Theorem

1. Be an infinitely long cylinder of radius a, placed along the axis $z'z$, containing a uniform charge density ρ.

   (a) Based on symmetry considerations due to the uniformity of the charge distribution, determine for an arbitrary point M in space

      i. The direction of $\vec{E}(M)$,

      ii. The set of coordinates on which $\vec{E}(M)$ depends.

   (b) Calculate the electrostatic field $\vec{E}(M)$ for $0 < \rho < \infty$ using Gauss's theorem!

   (c) From the preceding result deduce $V(M)$.

   (d) Let the radius a of the cylinder tend to zero. As a consequence, the charge distribution will become that of an infinitely thin and infinitely long wire, carrying the one-dimensional charge density λ. Use the result obtained in 1b to determine $\vec{E}(M)$ for this new charge distribution.

2. We consider an infinitely extended and infinitely thin sheet in the (y, z) plane, carrying a uniform and positive surface charge density σ.

   (a) Determine the direction of the electrostatic field at any point in space using symmetry arguments.

   (b) Calculate the electrostatic field $\vec{E}(M)$ using Gauss's Theorem.

   (c) Trace the graph of the scalar function $E(x)$.

Homework:

Consider two infinitely extended parallel planes P_1 and P_2 separated by d and placed at $x = d/2$ and $x = -d/2$, respectively. P_1 carries a uniform charge density σ_1, P_2 a uniform charge density σ_2. Determine the electrostatic field at any point M. Distinguish two different cases: 1) $\sigma_1 = \sigma_2 = \sigma$, and 2) $\sigma_1 = -\sigma_2 = \sigma$, with $\sigma > 0$. Trace the graph of the function $E(x)$.

.5 Exercises 5: Divergence and Green's Identities

1. Divergence

   (a) Be a position vector $\vec{r} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$. Calculate div $\vec{r}$.

   (b) Use the preceding result and Gauss-Ostrogradsky's Theorem to calculate the volume of a sphere of radius R.

   (c) Calculate div $\vec{e}_r$ with $\vec{e}_r$ the normal radial vector.

   (d) Be a vector field $\vec{A} = -y\vec{e}_x + x\vec{e}_y$. Visualize $\vec{A}$ in the plane (x, y). Calculate div$\vec{A}$.

2. Green's Theorems

   Be $\phi(x, y, z)$ and $\psi(x, y, z)$ arbitrary differentiable scalar fields. We can in general write a vector field as $\vec{A} = \phi\,\vec{\text{grad}}\psi$. The derivative with respect to a normal coordinate $\vec{n}$ onto a surface $S(\mathcal{V})$ delimiting the volume $\mathcal{V}$ be $\frac{\partial}{\partial n} = \vec{n} \cdot \vec{\text{grad}}$.

   (a) Deduce the first of Green's identities:

$$\iiint_{\mathcal{V}} \left[\phi\Delta\psi + \left(\vec{\text{grad}}\psi\right)\left(\vec{\text{grad}}\phi\right)\right] d\mathcal{V} = \oiint_{S(\mathcal{V})} \phi\frac{\partial\psi}{\partial n} dS \qquad (1)$$

   using the Gauss-Ostrogradsky theorem for the vector field $\vec{A}$.

   (b) Use this result to deduce Green's second identity

$$\iiint_{\mathcal{V}} \left[\phi\Delta\psi - \psi\Delta\phi\right] d\mathcal{V} = \oiint_{S(\mathcal{V})} \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right] dS \qquad (2)$$

.6 Exercises 6: Electrostatic Multipole Expansion

1. Prove the Taylor expansion in Eq. (1.98).

2. Legendre's Polynomials
   Use Rodriguez' formula to calculate the first polynomials up to $k = 3$ and verify that Eqs. (1.101) and (1.104) are indeed identical for this set.

3. A single point charge be located at the point (x, y, z) in cartesian coordinates. Find the monopole moment and the dipole moment for this system.

4. Determine the monopole moment and the dipole moment of the following distribution of point charges in the (x, y) plane:
   $-q$ in $(-2, 0)$, $-q$ in $(2, 0)$, $3q$ in $(0, 2)$, $q > 0$. Determine the dipole moment of the same distribution after having displaced the charges by $\vec{a}$ (which corresponds to a displacement of the origin).

5. Determine the monopole moment and the dipole moment of the following distribution of point charges in the (x, y) plane:
   q in $(-1, 1)$, $-q$ in $(-1, -1)$, $q > 0$.
   Determine the dipole moment of the same distribution after having displaced the charges by $\vec{a}$. Conclusion?

6. Point charges are placed at the corners of a cube of edge a. The charges and their locations are as follows: $-3q$ at $(0, 0, 0)$, $-2q$ at $(a, 0, 0)$, $-q$ at $(a, a, 0)$, q at $(0, a, 0)$, $2q$ at $(0, a, a)$, $3q$ at (a, a, a), $4q$ at $(a, 0, a)$, $5q$ at $(0, 0, a)$. Find the monopole moment and the dipole moment of this charge distribution.

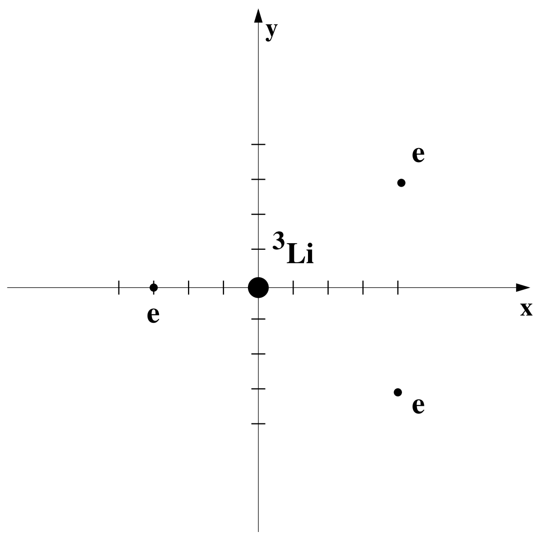
Homework:

1. *Be a sphere centered at the origin. Its upper hemisphere carries a uniform charge volume charge density ρ_-, its lower hemisphere a*

uniform charge volume charge density ρ_-, *with* $\rho_- = -\rho_+$. *Determine the electric dipole moment of the system.*

.7 Exercises 7: Multipoles, Electrostatic Energy, and Continuity Equation

1. Estimation of the ionization potential of a lithium (Li) atom.
   We use a simple electrostatic model of a light atom by placing point charges into the (x, y) plane in the following manner:



The nucleus of the Li atom carries charge Ze with $Z = 3$. The unit of length for the grid be unity.

Figure 1: Electrostatic model of the Li atom

   (a) Calculate the electric monopole moment of the charge distribution (what is the result if the lower right electron is removed?).

   (b) Calculate the electric dipole moment of the charge distribution with respect to the given origin (what is the result if the lower right electron is removed?).

   (c) Shift the origin to the point $(-1, -1)$. Recalculate the electric dipole moment (what is the result if the lower right electron is removed?).

   (d) Calculate the electrostatic energy of the configuration given in Figure 1 using an explicit expression for point charges. Give the result in units of $\frac{e^2}{4\pi\varepsilon_0}$.

(e) *Show that the same result is obtained if a surface charge density function $\sigma(x, y)$ is introduced, i.e., use the expression* $\varepsilon_e = \frac{1}{2} \iint_S \sigma V \, dS$.

(f) The ionization potential (IP)[2] is in this context defined as IP $= -\varepsilon_e$. Calculate IP in units of $[J]$ and $[eV]$ with the following values for the constants:

$$e \approx 1.602 \cdot 10^{-19} \, [C]$$
$$\varepsilon_0 \approx 8.854 \cdot 10^{-12} \, \left[\frac{s^2 \, C^2}{kg \, m^3}\right]$$
$$1[J] \approx 6.241 \cdot 10^{18} \, [eV]$$

The length unit for the grid is now taken to $10[pm]$ which is the typical length scale "inside" an atom.

2. At a given instant, a system has a charged current density given by $\vec{J} = A(x^3\vec{e}_x + y^3\vec{e}_y + z^3\vec{e}_z)$, where A is a positive constant.

(a) In what units will A be measured?

(b) At this instant, what is the change of the charge density at the point $(2, -1, 4)[m]$?

[2] This is the jargon of atomic physicists. It is more correct to call this ionization energy and to relate it to the ionization potential *via* the charge of the electron.

.8 Exercises 8: Lorentz Force and Biot and Savart's law

1. Be a point charge q moving with constant velocity v along the z axis in positive direction, in presence of a constant external electric field $\vec{E} = E_x\vec{e}_x + E_y\vec{e}_y + E_z\vec{e}_z$ a constant external magnetic field $\vec{B} = B_x\vec{e}_x + B_y\vec{e}_y + B_z\vec{e}_z$.
   Calculate the work that is carried out on the particle when it travels from O to point z. Conclusion?

2. Magnetic field created by a circular loop of wire.
   Be a circular loop of wire of radius R, centered around the Oz axis, carrying a stationary linear current intensity I.

   (a) Determine $\vec{B}(M)$ for any point M on the z axis.

   (b) Plot the graph of $B(z)$.

3. Magnetic field created by a straight wire.
   We consider a piece of rectilinear wire of length L, oriented along the Oz axis and carrying a stationary linear current intensity I in the direction $\vec{e}_z$.

   (a) Calculate the magnetic field $\vec{B}$ at a point M located at a distance ρ from the wire and in the plane orthogonal to the wire and cutting it in half.

   (b) Sketch a few magnetic field lines.

   (c) From the expression you have obtained, deduce the magnetic field for $\lim\limits_{L\to\infty}$.

.9 Exercises 9: Symmetry of the Magnetostatic Field and Ampère's Theorem

1. Rectilinear wire of infinite length.

   - Reconsider the wire from exercise section .8, exercise 3c. Use the symmetry properties of the magnetic field to determine the direction and the orientation of the magnetic field in the following points:

     $$(x, 0, 0) \text{ with } x > 0$$
     $$(0, y, 0) \text{ with } y > 0$$
     $$(0, 0, z)$$

   - Use Ampère's Theorem in integral form to determine the magnetic field $\vec{B}(M)$ for an arbitrary point M.

2. Reconsider the circular loop from exercise 2. Use the symmetry properties of the magnetic field to determine the direction and the orientation of the magnetic field in the following points:

   $$(0, 0, z) \text{ with } z \neq 0$$
   $$(0, 0, 0)$$
   $$(2R, 0, 0)$$

   Trace a few field lines of the resulting field $\vec{B}$.

Homework:

Suppose a rectilinear wire had the shape of a square, side length a. It carries a stationary current intensity I. Determine the magnetic field vector at the center of the square.

.10 Exercises 10: Curl, Ampère's Theorem in local form and Stokes' Theorem

1. Be a scalar field $V(\vec{r}) = \frac{f}{r}$ where f is a constant and r is a radial spherical coordinate. Calculate $\vec{\mathrm{rot}}\left(\vec{\mathrm{grad}} V(\vec{r})\right)$.

2. Prove the validity of Ampère's Theorem in local form for the case of the rectilinear wire for the points M in space where the magnetic field is well defined.

3. Be a vector field $\vec{A}(\vec{x}) = x_2\vec{e}_1 - x_1\vec{e}_2$. Use the (x_1, x_2) plane and a square contour to validate Stokes' Theorem for the given vector field $\vec{A}$.

.11 Exercises 11: Vector analysis in Magnetostatics and Local Structure Equation

1. Prove the validity of relation (3.38).
2. Prove the validity of relation (3.40).
3. Validate the structure equation of the magnetostatic field for the case of the field surrounding a rectilinear wire carrying a stationary linear current intensity I.

.12 Exercises 12: Vector analysis in Magnetostatics and the Magnetic Vector Potential

1. Prove Eq. (3.52).

2. Validate the identity $\sum_{i=1}^{3} \varepsilon_{ijk}\,\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$.

3. Use the expression from exercise 2 to prove the identity $\vec{\mathrm{rot}}\left(\vec{a}(\vec{r}) \times \vec{b}(\vec{r})\right)$.

4. For a spatially uniform magnetostatic field, the vector potential can be written as[3]
$$\vec{A}(\vec{r}) = \frac{1}{2}\,\vec{B} \times \vec{r}. \tag{3}$$
   (a) By using the relation from exercise 3, show that this vector potential indeed represents the magnetostatic field.
   (b) Prove that $\vec{A}(\vec{r})$ is divergence free.
   (c) Visualize $\vec{A}(\vec{r})$.

Homework:

1. *Calculate the double curl of a vector field, i.e.,* $\vec{\mathrm{rot}}\left(\vec{\mathrm{rot}}\,\vec{G}(\vec{r})\right)$*, and give the result in the most compact form.*

[3] This kind of field and vector potential is important in atomic physics in the context of the Zeeman effect.

.13 Exercises 13: Gauge transformation and Magnetic Multipoles

.14 Vector analysis. Important relations and proofs

.14.1 Basic relations

$$\begin{aligned}
\operatorname{div}\left[a(\vec{r})\,\vec{b}(\vec{r})\right] &= \sum_{i=1}^{3} \vec{e}_i \cdot \frac{\partial}{\partial x_i}\left[a(\vec{r}) \sum_{j=1}^{3} \vec{e}_j\, b_j(\vec{r})\right] \\
&= \sum_{i,j=1}^{3} \delta_{ij} \frac{\partial}{\partial x_i}\left[a(\vec{r})\, b_j(\vec{r})\right] \\
&= \sum_{i=1}^{3} \left[\frac{\partial a(\vec{r})}{\partial x_i}\, b_i(\vec{r}) + a(\vec{r})\, \frac{\partial b_i(\vec{r})}{\partial x_i}\right] \\
&= a(\vec{r})\operatorname{div}\vec{b}(\vec{r}) + \vec{b}(\vec{r}) \cdot \vec{\operatorname{grad}}\, a(\vec{r})
\end{aligned} \tag{4}$$

.14.2 Relations involving vector products

When more complicated mathematical expressions have to be handled, it is a powerful technique to express the vector product by using the Levi-Civita symbol, ε, defined as

$$\varepsilon_{ijk} = \left\{\begin{array}{l} +1 \text{ if } i,j,k \text{ even permutation} \\ -1 \text{ if } i,j,k \text{ odd permutation} \\ 0 \text{ in all other cases.} \end{array}\right\} \forall i,j,k \in \{1,2,3\} \tag{5}$$

A basic application of Eq. (5) is in the definition of the vector product:

$$\vec{a} \times \vec{b} = \sum_{i,j,k=1}^{3} \varepsilon_{ijk}\, a_i b_j \vec{e}_k \tag{6}$$

Similarly, the curl of a vector field can be expressed in a very compact form as

$$\vec{\operatorname{rot}}\vec{G}(\vec{x}) = \boldsymbol{\nabla}_{\vec{x}} \times \vec{G}(\vec{x}) = \sum_{i,j,k=1}^{3} \varepsilon_{ijk} \frac{\partial}{\partial x_i} G_j(\vec{x})\, \vec{e}_k \tag{7}$$

Suppose we are confronted with an expression, actually occurring frequently in electrodynamics, like div $\left[\vec{a}\times\vec{b}\right]$, where $\vec{a}=\vec{a}(\vec{r})$ and $\vec{b}=\vec{b}(\vec{r})$ are vector fields. With the help of Eq. (5) we can write the divergence of the vector product mentioned above in a very convenient, index-based fashion

$$\begin{aligned}
\text{div}\left[\vec{a}(\vec{r})\times\vec{b}(\vec{r})\right] &= \sum_{i=1}^{3}\vec{e}_i\cdot\frac{\partial}{\partial x_i}\left[\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\varepsilon_{jkl}\,a_k(\vec{r})\,b_l(\vec{r})\,\vec{e}_j\right] \\
&= \sum_{i,j,k,l=1}^{3}\vec{e}_i\cdot\varepsilon_{jkl}\left(\frac{\partial a_k}{\partial x_i}b_l+a_k\frac{\partial b_l}{\partial x_i}\right)\vec{e}_j \\
&= \sum_{i,j,k,l=1}^{3}\delta_{ij}\,\varepsilon_{jkl}\left(\frac{\partial a_k}{\partial x_i}b_l+a_k\frac{\partial b_l}{\partial x_i}\right) \\
&= \sum_{j,k,l=1}^{3}\varepsilon_{jkl}\,\frac{\partial a_k}{\partial x_j}b_l+\sum_{j,k,l=1}^{3}\varepsilon_{jkl}\,a_k\frac{\partial b_l}{\partial x_j} \\
&= \sum_{j,k,l=1}^{3}\varepsilon_{klj}\,\frac{\partial a_l}{\partial x_k}b_j+\sum_{j,k,l=1}^{3}\varepsilon_{kjl}\,a_j\frac{\partial b_l}{\partial x_k} \\
&= \sum_{i,j,k,l=1}^{3}\vec{e}_i\cdot\vec{e}_j\,\varepsilon_{klj}\,\frac{\partial a_l}{\partial x_k}b_i+\sum_{i,j,k,l=1}^{3}\vec{e}_i\cdot\vec{e}_j\,\varepsilon_{kjl}\,a_i\frac{\partial b_l}{\partial x_k} \\
&= \sum_i b_i\,\vec{e}_i\cdot\sum_{j,k,l=1}^{3}\varepsilon_{klj}\,\frac{\partial a_l}{\partial x_k}\,\vec{e}_j+\sum_i a_i\,\vec{e}_i\cdot\sum_{j,k,l=1}^{3}\varepsilon_{kjl}\,\frac{\partial b_l}{\partial x_k}\,\vec{e}_j \\
\text{div}\left[\vec{a}\times\vec{b}\right] &= \vec{b}\cdot\text{rot}\,\vec{a}-\vec{a}\cdot\text{rot}\,\vec{b} \qquad (8)
\end{aligned}$$

The curl of the gradient of an arbitrary scalar field vanishes:

$$\begin{aligned}
\vec{\mathrm{rot}}\left(\vec{\mathrm{grad}}\, f(\vec{x})\right) &= \sum_{i,j,k=1}^{3} \varepsilon_{ijk}\, \frac{\partial}{\partial x_i} \left(\sum_{l=1}^{3} \vec{e}_l\, \frac{\partial}{\partial x_l}\, f(\vec{x}) \right) \cdot \vec{e}_j\, \vec{e}_k \\
&= \sum_{i,j,k,l=1}^{3} \varepsilon_{ijk}\, \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_l}\, f(\vec{x}) \delta_{jl}\, \vec{e}_k \\
&= \sum_{i,j,k=1}^{3} \varepsilon_{ijk}\, \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}\, f(\vec{x})\, \vec{e}_k \\
&= \left(\frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_1} \right) f(\vec{x})\, \vec{e}_3 \\
&\quad \left(\frac{\partial}{\partial x_2} \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_3} \frac{\partial}{\partial x_2} \right) f(\vec{x})\, \vec{e}_1 \\
&\quad \left(\frac{\partial}{\partial x_3} \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3} \frac{\partial}{\partial x_1} \right) f(\vec{x})\, \vec{e}_2 \\
\vec{\mathrm{rot}}\left(\vec{\mathrm{grad}}\, f(\vec{x})\right) &= \vec{0} \qquad (9)
\end{aligned}$$

Divergence of the curl of an arbitrary vector field:

$$\begin{aligned}
\text{div}\left(\vec{\text{rot}}\vec{G}(\vec{r})\right) &= \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\vec{e}_i \cdot \left(\sum_{j,k,l=1}^{3} \varepsilon_{jkl}\,\frac{\partial}{\partial x_j} G_k(\vec{r})\,\vec{e}_l\right) \\
&= \sum_{i,j,k,l=1}^{3} \delta_{il}\,\varepsilon_{jkl}\,\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\,G_k(\vec{r}) \\
&= \sum_{i,j,k=1}^{3} \varepsilon_{jki}\,\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\,G_k(\vec{r}) \\
&= \sum_{i,j,k=1}^{3} \varepsilon_{ijk}\,\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\,G_k(\vec{r}) \\
&= 0 \qquad (10)
\end{aligned}$$

The last step is evident as for every term k in the sum there is a pair of terms with $i \neq j$ with opposite signs.

Double curl of an arbitrary vector field:

$$\begin{aligned}
\vec{\mathrm{rot}}\left(\vec{\mathrm{rot}}\,\vec{G}(\vec{x})\right) &= \sum_{i,j,k=1}^{3} \varepsilon_{ijk}\,\frac{\partial}{\partial x_i}\left(\sum_{l,m,n=1}^{3} \varepsilon_{lmn}\,\frac{\partial}{\partial x_l} G_m(\vec{x})\vec{e}_n\right)\cdot\vec{e}_j\,\vec{e}_k \\
&= \sum_{i,j,k,l,m,n=1}^{3} \varepsilon_{ijk}\,\varepsilon_{lmn}\,\delta_{jn}\,\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_l} G_m(\vec{x})\vec{e}_k \\
&= \sum_{i,j,k,l,m=1}^{3} \varepsilon_{jki}\,\varepsilon_{jlm}\,\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_l} G_m(\vec{x})\vec{e}_k \\
&= \sum_{i,k,l,m=1}^{3} \left(\delta_{kl}\delta_{im} - \delta_{km}\delta_{il}\right)\,\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_l} G_m(\vec{x})\vec{e}_k \\
&= \sum_{i,k=1}^{3} \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k} G_i(\vec{x})\vec{e}_k - \sum_{i,k=1}^{3} \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i} G_k(\vec{x})\vec{e}_k \\
&= \sum_{k=1}^{3} \frac{\partial}{\partial x_k}\,\vec{e}_k\,\left(\sum_{i=1}^{3} \frac{\partial}{\partial x_i} G_i(\vec{x})\right) - \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}\,\left(\sum_{k=1}^{3} G_k(\vec{x})\vec{e}_k\right) \\
&= \vec{\mathrm{grad}}\left(\mathrm{div}\,\vec{G}(\vec{x})\right) - \Delta\,\vec{G}(\vec{x}) \qquad (11)
\end{aligned}$$

Curl of the product of a scalar and a vector field:

$$\begin{aligned}
\vec{\text{rot}}\left(u(\vec{x})\,\vec{G}(\vec{x})\right) &= \sum_{i,j,k=1}^{3} \varepsilon_{ijk}\,\frac{\partial}{\partial x_i}\left(\sum_{l=1}^{3} u(\vec{x})\,G_l(\vec{x})\vec{e}_l\right)\cdot\vec{e}_j\,\vec{e}_k \\
&= \sum_{i,j,k,l=1}^{3} \varepsilon_{ijk}\,\delta_{jl}\,\left(u(\vec{x})\,\frac{\partial G_l(\vec{x})}{\partial x_i} + \frac{\partial u(\vec{x})}{\partial x_i}G_l(\vec{x})\right)\,\vec{e}_k \\
&= \sum_{i,j,k=1}^{3} \varepsilon_{ijk}\,\left(u(\vec{x})\,\frac{\partial G_j(\vec{x})}{\partial x_i} + \frac{\partial u(\vec{x})}{\partial x_i}G_j(\vec{x})\right)\,\vec{e}_k \\
&= u(\vec{x})\,\sum_{i,j,k=1}^{3} \varepsilon_{ijk}\,\frac{\partial}{\partial x_i}G_j(\vec{x})\,\vec{e}_k \\
&\qquad + \sum_{i,j,k=1}^{3} \varepsilon_{ijk}\,\left(\sum_{l=1}^{3}\vec{e}_l\,\frac{\partial u(\vec{x})}{\partial x_l}\right)\cdot\vec{e}_i\,G_j(\vec{x})\,\vec{e}_k \\
&= u(\vec{x})\,\vec{\text{rot}}\vec{G}(\vec{x}) + \left(\vec{\text{grad}}\,u(\vec{x})\right)\times\vec{G}(\vec{x}) \qquad (12)
\end{aligned}$$